On New Invariant Solutions of Generalized Fokker–Planck Equation

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(Received September 4, 2003)

Abstract The generalized one-dimensional Fokker–Planck equation is analyzed via potential symmetry method and the invariant solutions under potential symmetries are obtained. Among those solutions, some are new and first reported.

PACS numbers: 02.30.Jr, 11.30.-j, 02.70.Wz

Key words: Fokker–Planck equation, potential symmetry, invariant solutions, symbolic computation

1 Introduction

The construction of the exact solutions, such as the group-invariant solutions, for partial differential equation(s) is one of the most important and essential tasks in nonlinear science. Recent years there have been considerable developments in symmetry methods for differential equation(s) (DEs) as evidenced by the number of research papers devoted to the subject.1–3 Potential symmetries admitted by a given partial differential equation(s) can be used to construct the invariant solutions that cannot be obtained as invariant solutions of its classical symmetries. In essence, if a given partial differential equation \( R(x,u) \) with independent variables \( x \) and dependent variables \( u \) can be written in a conservative form, maybe one can find its potential symmetries by analyzing the Lie point symmetries of an associated auxiliary system \( S\{x,u,v\} \), which is obtained by introducing a potential \( v(x) \) as additional dependent variable. Any group \( G_S \) of Lie transformations admitted by the system \( S\{x,u,v\} \) induces a symmetry for \( R(x,u) \). When at least one of the generators of \( G_S \) depends explicitly on the potential, then the corresponding symmetry of \( R(x,u) \) is called potential symmetry, which is neither a point nor a generalized (Lie-Bäcklund) symmetry. These potential symmetries of \( R(x,u) \) being point symmetries of the system \( S\{x,u,v\} \) can be determined by Lie’s algorithm. This fact makes the potential symmetries very useful in looking for the invariant solutions of \( R(x,u) \) using a reduction method.

In this paper, we analyze the generalized one-dimensional Fokker–Planck (FP) equation that has been studied in Ref. [3] again via the potential symmetry method. However, the mistake at the very beginning of the paper inevitably leads to the wrong results thereafter. Next, with the assistance of symbolic manipulation, we not only correct the errors but also obtain some new invariant solutions to the generalized FP equation.

2 Potential Symmetries for Generalized FP Equation

Consider the generalized one-dimensional Fokker–Planck (FP) equation of the form,[4]

\[
\partial u / \partial t = \partial (pu_x + qu) / \partial x ,
\]

where \( x, t \) are reals, \( p \) and \( q \) are arbitrary smooth functions of \( x \).

Fokker–Planck equation and its generalizations occupy a central position in statistical physics.[3] It is the master equation for the Ornstein–Uhlenbeck process that with a proper rescaling is the only stationary Gaussian Markov process. It has wide-ranging applications in physical, biological, and sociological phenomena. As is written, equation (1) is already in a conserved form. By Pucci’s theorems,[3] it is easy to see that equation (1) indeed admits potential symmetries. Its associated auxiliary system \( S\{x,t,u,v\} \) is given by

\[
v_x = u, \quad v_t = pu_x + qu ,
\]

where the unknown function \( v(x,t) \) is potential. If \( u(x,t) \) satisfies Eq. (2), then \( u(x,t) \) solves Eq. (1).

Supposing that equation (2) admits a one-parameter (\( e \)) Lie group of point transformations,

\[
\begin{align*}
ax &= x + e\xi(x,t,u,v) + o(e^2) , \\
bt &= t + e\tau(x,t,u,v) + o(e^2) , \\
u^* &= u + e\eta(x,t,u,v) + o(e^2) , \\
v^* &= v + e\phi(x,t,u,v) + o(e^2) ,
\end{align*}
\]

and it is completely determined by the infinitesimals \( \xi, \tau, \eta, \) and \( \phi \). Denote the infinitesimal generator corresponding to Eq. (3) as

\[
X = \xi(x,t,u,v)(\partial / \partial x) + \tau(x,t,u,v)(\partial / \partial t) + \eta(x,t,u,v)(\partial / \partial u) + \phi(x,t,u,v)(\partial / \partial v) .
\]

Point symmetries, which verify \( \xi^2 + \tau^2 + \eta^2 = 0 \), correspond to point symmetries of Eq. (1). Instead, potential symmetries of Eq. (1) are obtained, if \( \xi^2 + \tau^2 + \eta^2 \neq 0 \).

According to the infinitesimal criterion for invariance of a system of PDE’s, we have

\[
X^{(1)}|_{\theta = 0} = 0 , \quad X^{(1)}|_{v = pu_x - qu} = 0 ,
\]

where \( X^{(1)} \) is the first prolongation of the infinitesimal generator (4), which is given explicitly in terms of \( \xi, \tau, \) and \( \phi \) (cf., Ref. [5]). Equations (5) become

\[
\begin{align*}
\phi_x + (\phi_u - \xi_v)uv_x - \xi v_x + \phi_uu_x - \tau_xv_x - \tau uv_xv_x - \eta = 0 , \\
\phi_t + (\phi_u - \tau_v)v_x - \tau v_x + \phi_uu_x - \tau uv_xv_x - \xi u_xv_x - \xi u_xv_x - \xi v_xv_x - \eta = (p'uv_x + q'uv_x)\xi .
\end{align*}
\]

The project supported by the Research Fund for the Doctoral Program of Higher Education of China under Grant No. 20020269003
Eliminate \( v_x \) and \( v_t \) through substitution \( v_x \) by \( u_x \), and \( v_t \) by \( pu_x + qu \) into Eq. (6), which are polynomial equations in the components of \( u_x, u_t \) and must hold for arbitrary values of these components. Consequently the coefficients of the polynomial equations must vanish separately, which result in a system of linear homogeneous partial differential equations, named determining equations for the infinitesimals \( \xi, \tau, \eta, \phi \). For the determining equations, according to the second theorem given in Ref. [2], we know \( \tau \) only depends on that \( t \), infinitesimal \( \xi \) is independent of \( u \) and \( v \), and \( \eta \) and \( \phi \) are linear in \( u \) and \( v \). Thus from Eq. (6) we get

\[
\tau = \tau(t), \quad \xi = \xi(x, t), \quad \phi(u) = 0, \\
\phi_x - \eta(\phi_v - \xi_u)u - \xi_vu^2 = 0, \\
p(\phi_v - \xi_u - \xi_t + p^2\xi) = 0, \\
\phi_t - \xi_x - \phi_x + \eta[q(\phi_v - \tau) - q'\xi - \xi_t - \eta]u = 0
\]

with

\[
\eta = f(x, t)u + g(x, t)v, \quad \phi = k(x, t)v, \quad f, g, \text{ and } k \text{ are arbitrary smooth functions of } x \text{ and } t.
\]

Then, after some algebra calculations we get the system of determining equations as

\[
\tau = \tau(t), \quad \xi = \xi(x, t), \quad \kappa_x - g = 0, \\
k - \xi_u - f = 0, \quad 2\xi_x - \tau_u - (p'/p)\xi = 0, \\
k\xi - p\xi_x - q\eta = 0, \quad g = (pq_x + qp)_x, \\
(p'/p - q'/q)\xi - \xi_x - (1/q)(\xi + pf_x + pq) = 0.
\]

We mention that for the last equations of Eqs. (8) and (9), the corresponding ones given in Ref. [3] both miss the function \( p \) in the last term, which inevitably leads to the wrong results thereafter.

3 Invariant Solutions for Generalized FP Equation

To construct the invariant solutions of Eq. (1) under potential symmetries, one should obtain the group-invariant solutions for Eq. (1) under the point symmetries. To reach this aim, one assumes that \( u = \Theta_1(x, t) \) and \( v = \Theta_2(x, t) \) are group-invariant solutions of Eq. (1). Then according to the invariant surface condition[6] we have

\[
\xi\partial\Theta_1/\partial x + \tau\partial\Theta_1/\partial t = \eta, \quad \xi\partial\Theta_2/\partial x + \tau\partial\Theta_2/\partial t = \phi.
\]

The general solution of Eq. (10) can be found by integrating the characteristic system

\[
dx/\xi = dt/\tau = d\Theta_1/\eta = d\Theta_2/\phi.
\]

The general solution of Eq. (11) can be written in the form

\[
\omega_1(x, t, u, v) = c_1, \quad \omega_2(x, t, u, v) = c_2,
\]

\[
\omega_3(x, t, u, v) = c_3,
\]

in which \( c_1, c_2, c_3 \) are constants of integration. If assuming \( c_1 = z \) as a similarity variable and \( c_2 = \zeta_1(z) \), \( c_3 = \zeta_2(z) \), from Eq. (12) we obtain

\[
u = U(x, t, z, \zeta_1(z), \zeta_2(z)), \quad (13)
\]

\[
u = V(x, t, z, \zeta_1(z), \zeta_2(z)), \quad (14)
\]

\[
\Psi(x, t, z, \zeta_1(z), \zeta_2(z)) = 0
\]

for some functions \( U, V, \text{ and } \Psi \). Functions \( \zeta_1(z) \) and \( \zeta_2(z) \) will be determined later. Equation (15) defines implicitly the similarity variable \( z \) as a function of \( x, t \).

To determine \( \zeta_1(z) \) and \( \zeta_2(z) \), on the one hand, one can solve an ordinary system \( \Omega \), which is obtained by substituting Eqs. (13) and (14) into Eq. (2). On the other hand, substituting Eq. (13) into Eq. (1), we obtain a relation involving \( z, \zeta_1, \zeta_2 \), and the derivatives of \( \zeta_1, \zeta_2 \) up to order \( k \), and one parameter given by \( x \) or \( t \). By imposing that the relation is identically zero for any value of the parameter, this will result in an ordinary system \( \Omega \) on \( \zeta_1(z) \). From \( \Omega \) and \( \tilde{\Omega} \) we can determine \( \zeta_1(z) \), \( \zeta_2(z) \) explicitly and then obtain the potential symmetry-invariant solutions of Eq. (1), which are generally not invariant solutions of any point symmetry admitted Eq. (1).

Next, under the various cases of \( p \) and \( q \) we compute the Lie symmetry group of Eq. (2) and then obtain the special invariant solutions of Eq. (1). From now on, we denote \( c_i (i = 1, 2, \ldots) \) and \( \alpha, \beta \) as arbitrary constants.

3.1 \( p = a \) and \( q = x \)

In this case, solving Eq. (9), a six-parameter \((c_1 \sim c_6)\) Lie group of transformations admitted by Eq. (1) is given:

\[
\tau = -ac_1 \exp(2t) + c_4 \exp(-2t) + c_5,
\]

\[
\xi = -ac_1 \exp(2t) - c_5 \exp(-2t) + c_2 \exp(t) + c_4 \exp(-2t) + c_6 u + 2c_3 \exp(-2t) - c_2 \exp(t),
\]

\[
\eta = [c_1(x^2 + 2a) \exp(2t) - c_2 x \exp(t) + c_4 \exp(-2t) + c_6 u] + [2c_3 \exp(-2t) - c_2 \exp(t)] v,
\]

\[
\phi = [c_1 x^2 \exp(2t) - c_2 x \exp(t) + ac_1 \exp(2t) + c_6 v].
\]

The corresponding infinitesimal generators of this group are

\[
X_1 = -a \exp(2t) \frac{\partial}{\partial t} - ax \exp(2t) \frac{\partial}{\partial x} + [(x^2 + 2a)u + 2xv] \exp(2t) \frac{\partial}{\partial v},
\]

\[
X_2 = a \exp(t) \frac{\partial}{\partial t} - (xu + v) \exp(t) \frac{\partial}{\partial x} - xv \exp(t) \frac{\partial}{\partial v},
\]

\[
X_3 = \exp(-t) \frac{\partial}{\partial x} + x \exp(-2t) \frac{\partial}{\partial u} - u \exp(-2t) \frac{\partial}{\partial v},
\]

\[
X_4 = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v},
\]

and \( x \)-dimensional symmetry. Among them, only \( X_1 \) and \( X_2 \) are potential symmetries.

For the potential symmetry \( X_1 \) and in the case of \( a = 1 \), we have the same results as given in Eq. (2); for the case of \( a \neq 1 \), the solution is trivial. For the potential symmetry \( X_2 \), same as before, solving the characteristic equations

\[
\frac{dx}{a \exp(t)} = \frac{du}{-(xu + v) \exp(t)} = \frac{dv}{-xv \exp(t)}, \quad dt = 0,
\]

we obtain three integrals

\[
\zeta_7 = t, \quad c_8 = (u + xv/a) \exp(x^2/2a),
\]

\[
c_9 = v \exp(x^2/2a).
\]
Then, the similarity variable is \( z = t \). Let \( c_8 = \zeta_1(z) \) and \( c_9 = \zeta_2(z) \), we obtain the invariant solutions \( u \) and \( v \) with the form

\[
\begin{align*}
    u &= \left[ \zeta_1(z) - (x/a)\zeta_2(z) \right] \exp(-x^2/2a), \\
    v &= \zeta_2(z) \exp(-x^2/2a).
\end{align*}
\]

Substituting the first equation of Eqs. (20) into Eq. (1) we obtain

\[
\zeta_1' - x(\zeta_2' + \zeta_2) = 0,
\]

which must hold for any value of \( x \), then we get the system \( \hat{O} \) as

\[
\zeta_1' = 0, \quad \zeta_2' + \zeta_2 = 0.
\]

Solving Eq. (22) yields

\[
\zeta_1(z) = c_{10}, \quad \zeta_2(z) = c_{11} \exp(-z).
\]

On the other hand, substituting Eq. (20) into Eq. (2) gives the system \( \hat{O} \)

\[
\exp(-x^2/2a)\zeta_1' = 0, \quad \exp(-x^2/2a)\zeta_2' + \zeta_2 = 0.
\]

Thus, we have \( \zeta_1(z) = 0 \) and \( \zeta_2(z) = \alpha \exp(-z) \). It is easy to see that \( c_{10} = 0 \) and \( c_{11} = \alpha \). Then the invariant solution of Eq. (1) is

\[
u(x, t) = -(tx/a) \exp(-t) \exp(-x^2/2a).
\]

This solution is new and first reported here, which is smooth and bound. It is easy to see that, for a constant diffusion and a linear drift \( x \), the distribution function \( u(x, t) \) plunges down with respect to \( x \) and attenuates exponentially with the change of \( t \).

### 3.2 \( p = a \) and \( q = b \)

In this case, solving Eq. (9), a five-parameter \((c_1 \sim c_5)\) Lie group of transformations admitted by Eq. (1) is given as

\[
\tau = 2bxt + c_1, \quad \xi = c_4x - (bc_4 + 2ac_2)t + c_5, \\
\eta = (x + bt)c_2 + c_3 - c_4u + c_2v, \\
\phi = (x + bt)c_2 + c_3v.
\]

The corresponding infinitesimal generators of this group are

\[
\begin{align*}
    X_1 &= \frac{\partial}{\partial t}, \\
    X_2 &= \frac{\partial}{\partial x}, \\
    X_3 &= u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, \\
    X_4 &= 2t \frac{\partial}{\partial t} + (x - bt) \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, \\
    X_5 &= -2a \frac{\partial}{\partial x} + [(x + bt)u + v] \frac{\partial}{\partial u} + (x + bt)v \frac{\partial}{\partial v},
\end{align*}
\]

and \( \infty \)-dimensional symmetry. Clearly, only \( X_5 \) is a potential symmetry for Eq. (1).

For the potential symmetry \( X_5 \), the characteristic equations related to the invariant surface conditions are

\[
2atx + (x + bt)u + v = 0, \quad 2atx + (x + bt)v = 0.
\]

Solving its characteristic equations,

\[
\begin{align*}
    dx &= \frac{du}{2at} = \frac{dv}{-(x + bt)v}, \\
    dt &= 0,
\end{align*}
\]

we obtain three integrals

\[
\begin{align*}
    c_6 &= t, \quad c_7 = v \exp\left[ (x/2a)/(x/2t + b) \right], \\
    c_8 &= u/v + x/2at.
\end{align*}
\]

If assuming \( \zeta_0 = z \) as the similarity variable, \( \zeta_7 = \zeta_1(z) \), and \( \zeta_8 = \zeta_2(z) \) as similarity functions in Eq. (30), we obtain the invariant solutions \( u \) and \( v \) with the form

\[
\begin{align*}
    u &= \zeta_2(z)[\zeta_1(z) - (x/2at)] \exp\left[ -(x/2a)(x/2t + b) \right], \\
    v &= \zeta_2(z) \exp\left[ -(x/2a)(x/2t + b) \right].
\end{align*}
\]

The first equation of Eq. (31) verifies Eq. (1), then we have

\[
\begin{align*}
    &-2\zeta_2a - 4\zeta_2a^2\zeta_1 - 2(4\zeta_2a^2\zeta_1^2 + \zeta_2\zeta_1^2a^2b^2) + 4\zeta_2a^2z^2\zeta_1 + 2\zeta_2a^2\zeta_1 = 0.
\end{align*}
\]

It is identical even without the restraint \( a = 1 \), whereas it is necessary in Ref. [3]. Because equation (32) must hold for any value of \( a \), we get the system \( \hat{O} \) as

\[
-2\zeta_2a - 4\zeta_2a^2\zeta_1 - 2b^2z = 0, \\
4\zeta_2a^2z^2\zeta_1 + \zeta_2\zeta_1^2a^2b^2 + 4\zeta_2a^2z^2\zeta_1 + 2\zeta_2a^2\zeta_1 = 0.
\]

Solving Eq. (33) yields

\[
\zeta_1(z) = c_9, \quad \zeta_2(z) = c_{10}t^{-1/2} \exp(-b^2t/4a).
\]

Substituting Eq. (31) into the auxiliary system (2) yields the system \( \hat{O} \) as

\[
\zeta_1(z) = -b/2a, \quad 4a\zeta_2^2 + (b^2z + 2a)\zeta_2 = 0,
\]

which on solving yields

\[
\zeta_1(z) = -b/2a, \quad \zeta_2(z) = \alpha^{-1/2} \exp(-b^2t/4a).
\]

Comparing Eq. (34) with Eq. (36) we have \( c_9 = -b/2a, c_{10} = \alpha \). Thus, the invariant solution of Eq. (1) is

\[
u(x, t) = -\frac{a}{2a} - t^{-3/2}(x + bt) \exp\left[ -(x + bt)^2/4at \right].
\]

It is, as far as we know, a type of new group-invariant solution to Eq. (1) and first reported here. Obviously, for constant diffusion and drift, the distribution function \( u(x, t) \) attenuates slower than the case in section with the change of \( t \).

### 3.3 \( p = a \) and \( q = 1/x \)

In this case, a four-parameter \((c_1 \sim c_4)\) Lie group of transformations admitted by Eq. (1) is given by

\[
\tau = -2ax^2 + 2bx + c_4, \quad \xi = -2ac_1xt + c_4x, \\
\eta = \left( \frac{x^2}{2} + t(3a + 1) \right)c_1 + c_2 - c_3 \right) u + c_1 x v, \\
\phi = \left( \frac{x^2}{2} + t(a + 1) \right)c_1 + c_2 \right) v.
\]

The corresponding infinitesimal generators of this group are

\[
\begin{align*}
    X_1 &= -2a^2 \frac{\partial}{\partial u} - 2ax \frac{\partial}{\partial x} + \left[ \left( \frac{x^2}{2} + t(3a + 1) \right) u + x v \right] \frac{\partial}{\partial u}, \\
    X_2 &= u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, \\
    X_3 &= 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, \\
    X_4 &= \phi = \frac{\partial}{\partial \phi}.
\end{align*}
\]

and \( \infty \)-dimensional symmetry, among which only \( X_1 \) is a potential symmetry. For the potential symmetry \( X_1 \), solving the characteristic equations, we obtain three integrals,

\[
\begin{align*}
    c_5 &= \frac{x}{t}, \quad c_6 = \exp\left( \frac{x^2}{4at} + \frac{x^2}{2at} \right) (u + \frac{vx}{2at}).
\end{align*}
\]
\[ c_7 = \exp\left(\frac{x^2}{4at}\right)x^{(a+1)/2a}v. \] (40)

Likewise, \( z = x/t \) is similarity variable. By setting
\[ c_0 = \zeta_1(z), \quad c_1 = \zeta_2(z), \]
we obtain the invariant solutions \( u \) and \( v \) with the form
\[ u = \left[\zeta_1(z) - \frac{x^2}{2at}\zeta_2(z)\right]x^{-3(a+1)/2a}\exp\left(-\frac{x^2}{4at}\right), \]
\[ v = \zeta_2(z)x^{-(a+1)/2a}\exp\left(-\frac{x^2}{4at}\right). \] (41)

Substituting the first equation of Eq. (41) into Eq. (1), we obtain
\[ xz[4a^2z^2\zeta_0'' + 4a^2z\zeta_0' - (1 + a^2 + 2a)\zeta_0] - 2[4a^3z^3\zeta_0'' - 2a^2z\zeta_0'] = 0, \] (42)
which must hold for any value of \( x \). Then we get the system \( \mathcal{O} \) as
\[ 4a^2z^2\zeta_0'' + 4a^2z\zeta_0' - (1 + a^2 + 2a)\zeta_0 - 8a^2\zeta_1 = 0, \]
\[ 4a^3z^3\zeta_0'' - 12a^2z\zeta_0' - (a - 15a^3 + 2a^2)\zeta_1 = 0. \] (43)

Solving Eq. (43), then when \( a \neq 1, -1/3 \), we obtain
\[ \zeta_1(z) = c_8 z^{3(a-1)/2a} + c_9 z^{5(a+1)/2a}, \]
\[ \zeta_2(z) = c_8 \frac{a}{a-1} z^{3(a-1)/2a} + c_9 \frac{a}{3a+1} z^{5(a+1)/2a} + c_{10} z^{(a+1)/2a} + c_{11} z^{-(a+1)/2a}. \] (44)

Substituting Eq. (41) into Eq. (2) gives the system \( \mathcal{O} \), one obtains
\[ 4az\zeta_1' - 6a\zeta_1 + 2\zeta_2 = 0, \]
\[ 2a\zeta_2' - \zeta_2 - 2a\zeta_1 = 0, \] (45)
which on solving yields
\[ \zeta_1(z) = \alpha z^{3(a-1)/2a}, \]
\[ \zeta_2(z) = \alpha \left( \frac{a}{a-1} z^{3(a-1)/2a} + \beta z^{(a+1)/2a} \right) \] (46)

Comparing Eq. (46) with Eq. (44) we have \( c_8 = \alpha, c_{10} = \beta, c_9 = c_{11} = 0. \) Thus, under the case of \( a \neq 1, -1/3 \), we indeed obtain such a type of group-invariant solution to Eq. (1) as
\[ u(x, t) = \left[\frac{\alpha x^{-1/4}x^{-(3a-1)/2a}}{-\frac{\alpha}{2(a-1)}x^{2(a-1)/4}x^{-(5a-1)/2a}} - \frac{\beta}{2a}x^{-(3a+1)/2a}\right]\exp\left(-\frac{x^2}{4at}\right). \] (47)

Next, we consider the special cases of \( a = 1 \) and \( a = -1/3 \), respectively.

For the case of \( a = 1 \), from Eq. (42) we get
\[ z^2\zeta_0'' - 3z\zeta_0' + 3\zeta_1 = 0, \quad z^2\zeta_0'' + z\zeta_0' - \zeta_2 - 2\zeta_1 = 0, \] (48)
which on solving yields
\[ \zeta_1(z) = c_1 z + c_2 z^{3}, \]
\[ \zeta_2(z) = c_1 z \ln(z) + \frac{1}{4} c_2 z^3 + c_3 z^{-1} + c_4 z. \] (49)

On the other hand, substituting \( a = 1 \) into Eq. (45) directly, we get
\[ \zeta_1(z) = 0, \quad \zeta_2(z) = \zeta_2(z) = 0, \] (50)
which on solving yields
\[ \zeta_1(z) = \alpha z, \quad \zeta_2(z) = \alpha z \ln(z) + \beta z. \] (51)
Comparing Eq. (49) with Eq. (51), we have \( c_1 = \alpha, c_2 = 0, \) \( c_3 = 0, c_4 = 0 \), then the group-invariant solution of Eq. (1) is
\[ u(x, t) = \left\{\left(\alpha x^{-1/4} - \frac{1}{2} x t^{-2}\left[\alpha \ln\left(\frac{x}{t}\right) + \beta\right]\right)\right\} \exp\left(-\frac{x^2}{4at}\right). \] (52)

For the case of \( a = -1/3 \), carrying through the same procedure as before, we get the group-invariant solution of Eq. (1) as
\[ u(x, t) = \left(\alpha x^{3/2} + \frac{3}{8} \alpha x^3 t^2 + \frac{3}{2} \beta x\right) \exp\left(\frac{3x^2}{4at}\right). \] (53)

The group-invariant solutions (47), (52), and (53) to Eq. (1), which, to our knowledge, are also new and first reported here.

4 Concluding Remarks

In this work, we introduce the new class of symmetries for the generalized one-dimensional FP equation via the potential symmetry method due to Bluman[1] and obtain its new special group-invariant solutions, which are neither point nor generalized symmetry invariant solutions. For all the so-called new invariant solutions given in this paper, we have verified them one by one by putting them back into Eq. (1) with the assistance of computer algebraic system Maple. We mention that for Eq. (1), Pucci et al.[2] have obtained its invariant solutions only in the case of \( p = 1, q = x \), however, they did not discuss the two cases of subsections 3.2 and 3.3. Succi et al.[7] have also obtained three kinds of similarity solutions in three special cases by using a direct algebraic method. It is easy to see that the three cases discussed respectively by Succi et al. and us are not completely the same and the solutions respectively obtained by Succi et al. and us do not overlap but are different from one another.

Acknowledgments

One of the authors (R.X. Yao) would like to express his sincere thanks to Prof. Fan En-Gui for offering the relevant references and for useful suggestions.

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