Painlevé property and conservation laws of multi-component mKdV equations

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Abstract

It is shown that two classes of $n$-component mKdV equations are Painlevé integrable, and the resonances occur, respectively, at $-1, 3, 4, 0, \ldots, 0, 1, \ldots, 1, 5, \ldots, 5$ for the geometric $n$-component mKdV equation which arises from curve motions in Euclidean space, and $-1, 3, 4, 0, \ldots, 0, 2, \ldots, 2, 4, \ldots, 4$ for another $n$-component mKdV equation which has an infinite number of Lie Bäcklund symmetries. It is also shown that the two-component geometric mKdV equation admits an infinite number of conservation laws. Conservation laws for several systems are constructed.

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1. Introduction

In this paper, we are mainly concerned with the Painlevé property and conservation laws of the geometric multi-component mKdV equation

$$\ddot{k}_t + \dot{k}_s\dot{k}_s + \frac{3}{2}k^2\ddot{k}_s = 0$$

(1)

and the following multi-component mKdV equation

$$\ddot{k}_t + \dot{k}_s\dot{k}_s + k^2\ddot{k}_s + (\ddot{k}, \dot{k}_s)\dddot{k} = 0,$$

(2)

where $\dddot{k} = (k_1, k_2, \ldots, k_n)$. When $k_i = 0$, $i = 2, \ldots, n$, both equations are reduced to the celebrated mKdV equation

$$k_t + k_s\dot{k}_s + \frac{3}{2}k^2k_s = 0,$$

(3)

which is integrable and its integrability was investigated by Wadati [1].
It is well known that system (1) arises from inextensible plane curves in Euclidean space [2,3]. In the plane case, the curve flow is governed by
\[ \gamma_t = -k_s n - \frac{1}{2} k^2 t, \]  
where $\gamma$ denotes the curve vector, $s$ and $k$ are, respectively, the arc-length and curvature of the curve, $n$ and $t$ are, respectively, the normal and tangent vectors. In the space case, the curve flow is specified by
\[ \gamma_t = -k_t b - k_s n - \frac{1}{2} k^2 t, \]  
where $n$, $b$ and $t$ are, respectively, the normal, binormal and tangent vectors, $k$ and $\tau$ are, respectively, curvature and torsion of the curve. It has been shown by Tsuchida and Wadati [4] that (1) is integrable, and also it has been proved in [5] that (1) and (2) have infinitely many Lie–Bäcklund symmetries. A natural question is that whether the systems (1) and (2) are integrable in other meanings. It is interesting to note that for the two-component mKdV equation (1), if the curve flow can be expressed as the graph $(x, u(x, t), v(x, t))$ of some function $u$ and $v$ on the $x$-axis. One finds that $u$ and $v$ satisfy [6]
\[
\begin{align*}
\frac{u_t}{u_{xx}} &= \left( \frac{1 + u^2_s + v^2_s}{(1 + u^2_s + v^2_s)^2} \right)^{1/2}, \\
\frac{v_t}{v_{xx}} &= \left( \frac{1 + u^2_s + v^2_s}{(1 + u^2_s + v^2_s)^2} \right)^{1/2}.
\end{align*}
\]

It is a generalization of the WKI equation [7], and we say it the two-component WKI equation. In fact, it turns out that the integrability of (6) is established by Qu et al. (6) who showed that it is the compatibility condition of a certain WKI scheme of inverse scattering transformation. Indeed, this WKI scheme for $(u, v)$ is connected to the AKNS scheme for $(k, \tau)$ by a gauge transformation.

It has been shown that the multi-component mKdV equation (1) is integrable in the sense that it has infinitely many higher order Lie–Bäcklund symmetries [5], where $\vec{k} = (k_1, k_2, \ldots, k_n)$ is a vector. Indeed, it has the following recursion operator which maps a symmetry to a new symmetry [8]:
\[
\mathcal{R} = D_j^2 + \vec{\kappa}^2 + \vec{\kappa} D_j^{-1} \langle \vec{\kappa}, \cdot \rangle - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (J_i \vec{k}) D_j^{-1} \langle J_i \vec{k}, \cdot \rangle,
\]
where $J_i$ are anti-symmetric matrices with nonzero entry of $(i, j)$ being 1 if $i < j$, that is $(J_{ij})_{kl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{kj}$.

Painlevé property and the existence of an infinite number of conservation laws are two important features to characterize integrability of nonlinear partial differential equations. Almost integrable equations which can be solved in terms of inverse scattering method are Painlevé integrable and have an infinite number of conservation laws [9–16]. We say that a system has Painlevé property if the solutions of the system are single-valued about a movable singularity manifold. According to Weiss and coworkers [9–12], to show (1) and (2) are Painlevé integrable, we may expand their solutions about a singularity manifold $\phi = 0$ as
\[ \vec{k} = \sum_{j=0}^{\infty} \vec{k}_j \phi^{j+\alpha}. \]

The substitution of (7) into (1) and (2) leads to conditions on $\alpha$ and recursion relation for the functions $\vec{k}_j$. If $\alpha$ is negative and the recursion relation are consistent, then we say the systems are integrable.

The purpose of this paper is to study the Painlevé property and conservation laws of systems (1) and (2). In Sections 2 and 3, we discuss the Painlevé property of systems (1) and (2). In Section 4, we obtain a gauge transformation which maps the multi-component WKI equation to system (1). The conservation laws of systems (1) and (2) are discussed in Section 5. Section 6 is a concluding remarks on this work.

2. Painlevé property of system (1)

To show (1) is Painlevé integrable, substituting $\vec{k} \sim \vec{k}_j \phi^j$ into (1) and the leading order analysis implies $\alpha = -1$. For $n = 2$, we obtain the recursion relation for the coefficients $k_{1j}$ and $k_{2j}$.
Therefore the resonances occur at $j = -1, 3, 4, 0, 1, 5$. The resonance at $j = -1$ corresponds to that the singular manifold $\phi = 0$ is arbitrary. If system (1) is Painlevé integrable, the conditions $f_1^j f_2^j = 0$, $j = 0, 1, 3, 4, 5$, are satisfied identically so that some functions $k_{j}^{k}$, $j = 0, 1, 3, 4, 5$, can be chosen arbitrary. With the Kruskal’s approach, one may choose $\phi(x, t) = x + \psi(t)$, where $\psi(t)$ is an arbitrary function of $t$. After a lengthy computation, one obtains

$$u_0(t) = -\sqrt{v_0^2 + 6}, \quad v_0 = v_0(t),$$

$$v_1(t) = \frac{\sqrt{v_0^2 + 6u_1}}{v_0}, \quad u_1 = u_1(t),$$

$$v_2(t) = \frac{1}{6} v_0^2 \psi' + 6u_2^2, \quad u_2 = -\frac{1}{6} \sqrt{v_0^2 + 6(v_0^2 \psi' + 6u_2^2)} - 1,$$

$$u_3 = u_3(t), \quad v_3(t) = -\frac{1}{2} v_0(t) \sqrt{v_0^2 + 6u_3(t)} + v_0(t),$$

$$u_4 = u_4(t), \quad v_4(t) = -\frac{1}{2} v_0^2(t) \sqrt{v_0^2 + 6u_4(t)} + v_0(t) u'_1(t) - u_1(t) v'_0(t),$$

$$u_5 = u_5(t), \quad v_5(t) = \frac{1}{12} \left[ 6v_0^2 \psi'' + v_0^4 \psi'' + 36v_0^2 \psi'' + v_0^4 \psi'' - 12 \sqrt{v_0^2 + 6(6u_5 + v_0^2 u_5 + v_0^4 \psi' u_3 - 6v_0^2 u_4)} - 6(v_0 + 1) v_0 u_1 u_1' \right] \sqrt{v_0^2(t) + 6},$$

where $v_0, u_1, u_3, u_4$ and $u_5$ are arbitrary functions of $t$. The substitution of (9) into the recursion relations implies that the recursion relations are satisfied identically. Therefore the system (1) with $n = 2$ is Painlevé integrable.

Similarly, for $n = 3$, $\tilde{k} = (k_1, k_2, k_3)$, we obtain the recursion relation replaced (8) by

$$(j + 1)(j - 3)(j - 4) f_3^j (j - 1)^2 (j - 5)^2 \left( \begin{array}{c} k_{1j} \\ k_{2j} \\ k_{3j} \end{array} \right) = \left( \begin{array}{c} f_1^j (\phi, \phi_x, \phi_{xx}, \tilde{k}_0, \tilde{k}_1, \ldots, \tilde{k}_{j-1}) \\ f_2^j (\phi, \phi_x, \phi_{xx}, \tilde{k}_0, \tilde{k}_1, \ldots, \tilde{k}_{j-1}) \\ f_3^j (\phi, \phi_x, \phi_{xx}, \tilde{k}_0, \tilde{k}_1, \ldots, \tilde{k}_{j-1}) \end{array} \right).$$

So the resonances occur at $j = -1, 3, 4, 0, 1, 1, 5, 5$. In general, for arbitrary $n$, we have

$$(j + 1)(j - 3)(j - 4)^{n-1} (j - 1)^{n-1} (j - 5)^{n-1} \left( \begin{array}{c} k_{1j} \\ k_{2j} \\ \vdots \\ k_{nj} \end{array} \right) = \left( \begin{array}{c} f_1^j (\phi, \phi_x, \phi_{xx}, \tilde{k}_0, \tilde{k}_1, \ldots, \tilde{k}_{j-1}) \\ f_2^j (\phi, \phi_x, \phi_{xx}, \tilde{k}_0, \tilde{k}_1, \ldots, \tilde{k}_{j-1}) \\ \vdots \\ f_n^j (\phi, \phi_x, \phi_{xx}, \tilde{k}_0, \tilde{k}_1, \ldots, \tilde{k}_{j-1}) \end{array} \right).$$

So the resonances occur at $j = -1, 3, 4, 0, \ldots, 0, 1, \ldots, 1, 5, \ldots, 5$. In the same vein, we can show that the geometric multi-component mKdV equation (1) is Painlevé integrable.

3. Painlevé property of system (2)

It has been shown in [5] that Eq. (2) has an infinite number of Lie–Bäcklund symmetries. We now show that the system is also Painlevé integrable. For $n = 2$, substituting (7) into (2), we obtain the recursion relation

$$(j + 1)(j - 3)(j - 4)(j - 2)(j - 4) \left( \begin{array}{c} k_{1j} \\ k_{2j} \end{array} \right) = \left( \begin{array}{c} f_1^j (\phi, \phi_x, \phi_{xx}, \tilde{k}_0, \tilde{k}_1, \ldots, \tilde{k}_{j-1}) \\ f_2^j (\phi, \phi_x, \phi_{xx}, \tilde{k}_0, \tilde{k}_1, \ldots, \tilde{k}_{j-1}) \end{array} \right).$$

So the resonances occur at $j = -1, 3, 4, 0, 2, 4$. With the Kruskal’s approach, without loss of generality, we use the ansatz $\phi(x, t) = x + \psi(t)$, and the coefficients $\tilde{k}_j$ in the expression (7) do not depend on $x$. Then the coefficients at the resonance points are given by
\[ u_0 = \sqrt{v_0^2 + 3i}, \quad v_0 = v_0(t), \]
\[ v_1 = u_1 = 0, \]
\[ v_2 = \frac{1}{2} \psi'(t) - 2\sqrt{v_0^2 + 3}u_2i, \quad u_2 = u_2(t), \]
\[ v_3 = \frac{(3v_3 - v_0^2 + v_0^2v_3)\sqrt{v_0^2 + 3}}{v_0(v_0^2 + 3)}, \quad v_3 = v_3(t), \]
\[ u_4 = u_4(t), \quad v_4 = v_4(t), \]

where \( u_0, u_2, v_3, u_4 \) and \( v_4 \) are arbitrary functions of \( t \), which means the two-component system (2) is Painlevé integrable.

For \( n = 3 \), the recursion relation for the coefficients \( k_j \) reads
\[
(j + 1)(j - 3)(j - 4)^f(j - 2)^2(j - 4)^2 \begin{pmatrix} k_{ij} \\ k_{2j} \\ k_{3j} \end{pmatrix} = \begin{pmatrix} f_j^1(\phi, \phi_t, \phi_{tt}, \bar{k}_0, \bar{k}_1, \ldots, \bar{k}_{j-1}) \\ f_j^2(\phi, \phi_t, \phi_{tt}, \bar{k}_0, \bar{k}_1, \ldots, \bar{k}_{j-1}) \\ f_j^3(\phi, \phi_t, \phi_{tt}, \bar{k}_0, \bar{k}_1, \ldots, \bar{k}_{j-1}) \end{pmatrix} .
\]

Hence the resonances occur at \( j = -1, 3, 4, 0, 2, 2, 4, 4 \). In general, for arbitrary \( n \), we obtain the recursion relation
\[
(j + 1)(j - 3)(j - 4)^{n-1}(j - 2)^n(j - 4)^n \begin{pmatrix} k_{ij} \\ k_{2j} \\ \vdots \\ k_{nj} \end{pmatrix} = \begin{pmatrix} f_j^1(\phi, \phi_t, \phi_{tt}, \bar{k}_0, \bar{k}_1, \ldots, \bar{k}_{j-1}) \\ f_j^2(\phi, \phi_t, \phi_{tt}, \bar{k}_0, \bar{k}_1, \ldots, \bar{k}_{j-1}) \\ \vdots \\ f_j^n(\phi, \phi_t, \phi_{tt}, \bar{k}_0, \bar{k}_1, \ldots, \bar{k}_{j-1}) \end{pmatrix} .
\]

So the resonances occur at \( j = -1, 3, 4, 0, \ldots, 0, 2, 2, 2, 4, 4, 4 \). Since the expressions for the coefficients are complicated, we omit it here. Therefore, the system (2) is Painlevé integrable.

4. **Gauge transformation for the system (1)**

It has been proved in [17,18] that the mKdV equation (3) is gauge equivalent to the WKI equation
\[
u_t = \left[ \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \right]_x .
\]

In other words, if the curvature \( k(x,t) \) of a plane curve satisfies the mKdV equation (3), then the corresponding local graph \( (x, u(x,t)) \) of the curve satisfies the WKI equation (15). It is also equivalent to the plane curve flow [11,18,19]
\[
\gamma_t = -k_n \mathbf{n} - \frac{1}{2} k^2 \mathbf{t},
\]
where \( s \) is the arc-length, \( \mathbf{n} \) and \( \mathbf{t} \) are, respectively, the normal and tangent vectors of the curve. Similarly, the two-component mKdV equation (1) is gauge equivalent to the two-component WKI equation
\[
u_t + \left[ \frac{u_{xx}}{(1 + u_x^2 + v_x^2)^{3/2}} \right] = 0 ,
\]
\[
u_t + \left[ \frac{v_{xx}}{(1 + u_x^2 + v_x^2)^{3/2}} \right] = 0
\]
and the corresponding space curve flow is
\[
\gamma_t = -k_n \mathbf{n} - k \mathbf{b} - \frac{1}{2} k^2 \mathbf{t},
\]
where \( s \) is the arc-length, \( \mathbf{n}, \mathbf{b} \) and \( \mathbf{t} \) are, respectively, the normal, binormal and tangent vectors of the space curve. In general, the \( n \)-component mKdV system (1) is geometrically equivalent to the \( n \)-component WKI equation.
\[
\begin{align*}
 u_{1t} + \frac{u_{1xx}}{(1 + \sum_{n=1}^{\infty} u_{nxx}^{2})^{3/2}} &= 0, \\
 u_{2t} + \frac{u_{2xx}}{(1 + \sum_{n=1}^{\infty} u_{nxx}^{2})^{3/2}} &= 0, \\
 \cdots
 u_{nt} + \frac{u_{nxx}}{(1 + \sum_{n=1}^{\infty} u_{nxx}^{2})^{3/2}} &= 0.
\end{align*}
\]

(19)

5. Conservation laws of the systems

In paper [6], we have shown that the two-component WKI equation (17) has an infinite number of conservation laws. As mentioned in Section 4, the two-component WKI equation is gauge equivalent to the two-component mKdV equation (1) via the gauge transformation

\[
\begin{align*}
 k_1 &= \sqrt{\frac{u_{ax}^2 + v_{ax}^2 + (u_x v_{ax} - v_x u_{ax})^2}{(1 + u_x^2 + v_x^2)^{3/2}}} \cos \varphi, \\
 k_2 &= \sqrt{\frac{u_{ax}^2 + v_{ax}^2 + (u_x v_{ax} - v_x u_{ax})^2}{(1 + u_x^2 + v_x^2)^{3/2}}} \sin \varphi,
\end{align*}
\]

where

\[
\varphi = \int_0^x \frac{u_{ax} v_{axx} - v_{ax} u_{axx}}{u_{ax}^2 + v_{ax}^2 + (u_x v_{ax} - v_x u_{ax})} \sqrt{1 + u_x^2 + v_x^2} \, dx.
\]

Using such equivalence, we can show that the two-component mKdV equation admits an infinite number of conservation laws and the conserved densities of the first several conserved laws are given by

\[D_1 = k_1^2 + k_2^2,\]
\[D_2 = k_1 k_{2x},\]
\[D_3 = (k_1^2 + k_2^2)^2 - 4(k_1^2 + k_2^2),\]
\[D_4 = 3k_1 k_2^2 k_{2x} + k_1^3 k_{2x} - 3k_2 k_{1xx},\]
\[D_5 = (k_1^2 + k_2^2)^3 - 12(k_1^2 k_2^2 + k_1^2 k_{2x}^2) - 20(k_1^2 k_{2x}^2 + k_{2x}^2 k_{1xx}) + 8(k_{1xx}^2 + k_{2xx}^2) - 16(k_{1xx} k_{2xx} - 16k_1 k_2 k_{1x} k_{2x}),\]
\[D_6 = \frac{5}{2}(k_1^2 k_1 k_{2xx} + k_1^2 k_2 k_{2x} - k_1 k_2 k_2^2 - k_2^2 k_{1xx}) - \frac{5}{4} k_1^2 k_2 k_{2x} + k_{1xx} k_{2xx} + \frac{15}{8} (k_1 k_3 k_{2xx} - k_1 k_2 k_{3xx}),\]
\[D_7 = 36(k_1^4 k_{2x}^2 + k_1^2 k_{2xx}^2) + (k_1^2 + k_2^2)^4 + 84(k_1^4 k_{2x}^2 + k_1^2 k_{2xx}^2) + \frac{288}{5} k_1 k_2 k_{1xx} k_{2xx} + 96 k_1^2 k_1 k_2 k_{1xx} + 120 k_1^2 k_2 k_{2xx}^2 - \frac{252}{5} (k_1^4 + k_1^2) + \frac{504}{5} (k_1^2 k_{2xx}^2 + k_1^2 k_{2xx}^2) + 72 (k_1^2 k_{2xx}^2 + k_1^2 k_{2xx}^2) + \frac{216}{5} (k_{1xx}^2 + k_{2xx}^2) + \frac{864}{5} k_2 k_{1xx} (k_{1xx} + k_{2xx}) - 48 k_1^2 k_{1xx} - 24 k_1^2 k_2 k_{1xx}^2 + \frac{72}{5} k_1^2 k_{2xx}.
\]

For the three-component mKdV system (1) with \(\vec{k} = (k_1, k_2, k_3)\), we obtain the first several conserved densities given by

\[D_1 = k_1^2 + k_2^2 + k_3^2,\]
\[D_2 = k_1 k_{3x} + k_2 k_{2x} + k_2 k_{1x},\]
\[D_3 = (k_1^2 + k_2^2 + k_3^2)^2 - 4(k_1^2 + k_2^2 + k_3^2) - k_1 k_2 k_{2x} + k_2 k_3 k_{1x} - 2(k_1 k_3 k_{2x} + k_2 k_3 k_{1x}) + k_1^2 k_{3xx} + k_2^2 k_{1xx} + k_1^2 k_{2xx} - k_3^2 k_{2xx},\]
For the three-component mKdV system (2) with \( \vec{k} = (k_1, k_2) \), we obtain the first several conserved densities as follows:

\[
D_1 = k_1^2 + k_2^2 + k_3^2
\]
\[
D_2 = (k_1^2 + k_2^2)^3 - 3(k_1^2 + k_2^2)
\]
\[
D_3 = -\frac{9}{2}(k_1^2 + k_2^2) + (k_1^2 + k_2^2)^3 - 15(k_1^2 + k_2^2) + 9k_1k_2k_3 - 6k_1^2k_3^2 + 3k_1^2k_2^2
\]
\[
D_4 = -\frac{63}{5}(k_1^2 + k_2^2) + (k_1^2 + k_2^2)^4 - 9(k_1^2k_2^2 + k_2^2k_3^2) - 42(k_1^2k_2^2 + k_2^2k_3^2) - 66k_1k_2k_3 - 51k_1^2k_2^2 + 48k_1^2k_2^2
\]
\[-\frac{126}{5}(k_1^2k_2^2 + k_2^2k_3^2) - 18k_1k_2k_3 - \frac{144}{5}k_1k_2k_3 - \frac{27}{5}(k_1^2k_2^2 + k_2^2k_3^2) + \frac{54}{5}
\]
\[
\times (k_1^2k_2^2 + k_2^2k_3^2) - \frac{117}{5}k_1^2k_2^2
\]
\[
D_5 = (k_1^2 + k_2^2)^3 + 18(k_1^2k_2^2 + k_2^2k_3^2) - 252k_1k_2k_3 - 216k_1^2k_2^2 - 126(k_1^2k_2^2 + k_1k_2k_3 - k_1k_2k_3)
\]
\[-k_1k_2k_3 + 27k_1^2k_2^2 + 189k_1^2k_2^2 + 99k_1^2k_2^2(k_1^2 + k_2^2) - \frac{1188}{7}k_1k_2k_3 - \frac{540}{7}k_1k_3
\]
\[-45(k_1^2k_2^2 + k_1^2k_3^2) - \frac{243}{7}(k_1^2k_2^2 + k_1^2k_3^2) - 116k_1k_2k_3 - \frac{54k_2k_3}{k_1k_2k_3}
\]
\[-312k_1k_2k_3 - 2(k_1k_2k_3 + k_1k_2k_3 + k_1k_2k_3) + \frac{1377}{7}(k_1^2k_2^2 + k_2^2k_3^2) + \frac{486}{7}k_2^2k_3^2 + 81(k_1^2k_3^2 + k_1^2k_2^2)
\]
\[-171(k_1^2k_2^2 + k_1^2k_3^2) - 114(k_1^2k_2^2 + k_1^2k_3^2) - \frac{324}{7}k_2k_3 - \frac{1620}{7}k_1k_2k_3 - \frac{108}{7}(k_1^2k_2^2 + k_1^2k_3^2)
\]
\[-92k_1^2k_2^2 + k_1^2k_3^2 - 12(k_1^2k_2^2 + k_1^2k_3^2) - 192(k_1^2k_2^2 + k_1^2k_3^2) + \frac{81}{14}(k_1^2k_2^2 + k_1^2k_3^2)
\]
\[-\frac{162}{7}k_2k_3 - \frac{270}{7}k_1k_2k_3
\]

For the three-component mKdV system (2) with \( \vec{k} = (k_1, k_2, k_3) \), we derive the first several conserved densities

\[
D_1 = k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_5^2 + k_6^2 + k_7^2 + k_8^2 + k_9^2
\]
\[
D_2 = (k_1^2 + k_2^2)^3 - 3(k_1^2 + k_2^2)
\]
\[
D_3 = -\frac{9}{2}(k_1^2 + k_2^2) + (k_1^2 + k_2^2)^3 - 15(k_1^2 + k_2^2) + 9k_1k_2k_3 - 6k_1^2k_3^2 + 3k_1^2k_2^2
\]
\[
D_4 = -\frac{63}{5}(k_1^2 + k_2^2) + (k_1^2 + k_2^2)^4 - 9(k_1^2k_2^2 + k_2^2k_3^2) - 42(k_1^2k_2^2 + k_2^2k_3^2) - 66k_1k_2k_3 - 51k_1^2k_2^2 + 48k_1^2k_2^2
\]
\[-\frac{126}{5}(k_1^2k_2^2 + k_2^2k_3^2) - 18k_1k_2k_3 - \frac{144}{5}k_1k_2k_3 - \frac{27}{5}(k_1^2k_2^2 + k_2^2k_3^2) + \frac{54}{5}
\]
\[
\times (k_1^2k_2^2 + k_2^2k_3^2) - \frac{117}{5}k_1^2k_2^2
\]
\[
D_5 = (k_1^2 + k_2^2)^3 + 18(k_1^2k_2^2 + k_2^2k_3^2) - 252k_1k_2k_3 - 216k_1^2k_2^2 - 126(k_1^2k_2^2 + k_1k_2k_3 - k_1k_2k_3)
\]
\[-k_1k_2k_3 + 27k_1^2k_2^2 + 189k_1^2k_2^2 + 99k_1^2k_2^2(k_1^2 + k_2^2) - \frac{1188}{7}k_1k_2k_3 - \frac{540}{7}k_1k_3
\]
\[-45(k_1^2k_2^2 + k_1^2k_3^2) - \frac{243}{7}(k_1^2k_2^2 + k_1^2k_3^2) - 116k_1k_2k_3 - \frac{54k_2k_3}{k_1k_2k_3}
\]
\[-312k_1k_2k_3 - 2(k_1k_2k_3 + k_1k_2k_3 + k_1k_2k_3) + \frac{1377}{7}(k_1^2k_2^2 + k_2^2k_3^2) + \frac{486}{7}k_2^2k_3^2 + 81(k_1^2k_3^2 + k_1^2k_2^2)
\]
\[-171(k_1^2k_2^2 + k_1^2k_3^2) - 114(k_1^2k_2^2 + k_1^2k_3^2) - \frac{324}{7}k_2k_3 - \frac{1620}{7}k_1k_2k_3 - \frac{108}{7}(k_1^2k_2^2 + k_1^2k_3^2)
\]
\[-92k_1^2k_2^2 + k_1^2k_3^2 - 12(k_1^2k_2^2 + k_1^2k_3^2) - 192(k_1^2k_2^2 + k_1^2k_3^2) + \frac{81}{14}(k_1^2k_2^2 + k_1^2k_3^2)
\]
\[-\frac{162}{7}k_2k_3 - \frac{270}{7}k_1k_2k_3
\]
6. Concluding remarks

In [20,21], it has been shown that the $K(m+2,m)$ model
\[ k_t + (k^m)_{xx} + \frac{m}{m+1} (k^{m+2})_x = 0, \] (21)
arises from the plane curve motions in certain Klein geometries, and it is geometrically equivalent to the generalized WKI equation [22,23]
\[ u_t + \left[ \frac{u_x}{(1 + u_x^2)^{3/2}} \right]_x = 0. \] (22)

Similarly, we can obtain the following multi-component $K(m+2,m)$ model
\[ k_{1t} + (k^m_{1})_{xx} + m \left[ k_{1}^{m-1} \sum_{i=1}^{n} k_{1i}^2 + \frac{1}{m+1} \sum_{i=1}^{n} k_{1i}^{m+1} \right] k_{1x} = 0, \]
\[ k_{2t} + (k^m_{2})_{xx} + m \left[ k_{2}^{m-1} \sum_{i=1}^{n} k_{2i}^2 + \frac{1}{m+1} \sum_{i=1}^{n} k_{2i}^{m+1} \right] k_{2x} = 0, \]
\[ \ldots, \]
\[ k_{nt} + (k^m_{n})_{xx} + m \left[ k_{n}^{m-1} \sum_{i=1}^{n} k_{ni}^2 + \frac{1}{m+1} \sum_{i=1}^{n} k_{ni}^{m+1} \right] k_{nx} = 0, \] (23)
from motion of inextensible space curves in Euclidean geometry. It is geometrically equivalent to the generalized multi-component WKI equation

\[ u_{1t} + \left[ \frac{u_{1xx}}{(1 + \sum_{i=1}^{n} u_{1i}^2)^{3/2}} \right]_x = 0, \]
\[ u_{2t} + \left[ \frac{u_{2xx}}{(1 + \sum_{i=1}^{n} u_{2i}^2)^{3/2}} \right]_x = 0, \]
\[ \ldots \]
\[ u_{nt} + \left[ \frac{u_{nxx}}{(1 + \sum_{i=1}^{n} u_{ni}^2)^{3/2}} \right]_x = 0. \] (24)

Lou and Wu [24] have shown that the $K(m+2,m)$ model is Painlevé integrable. In the same manner, we may show that the multi-component $K(m+2,m)$ model is also Painlevé integrable.

Rosenau [25] has shown that the $K(m+2,m)$ model admits compacton, it will be interesting to investigate whether the multi-component $K(m+2,m)$ model admits soliton solutions with compact support.

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