On $t/x$-Dependent Conservation Laws of the Generalized $n$th-Order KdV Equation

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By means of a direct algebraic method, new $(n-1)$th-order conservation laws that depend not only on $u$ and its derivatives but also explicitly on $t/x$ are constructed for the $n$th-order KdV equations.

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I. INTRODUCTION

As is well known, the classical $n$th-order ($n = 3, 5, 7, 9$) KdV equations have infinitely many conservation laws; most of them are polynomials that only depend on $u$ ($u = u(x,t)$) and its derivatives, and do not depend explicitly on $t/x$. The classical KdV equations, as a matter of fact, do have such conservation laws that depend explicitly on $t/x$. The aim of this paper is to construct the $(n-1)$th-order (highest derivative) $t/x$-dependent conservation laws for the $n$th-order KdV equations. To reach this end, we combine the direct algebraic method described below and the method presented in [1] with the assistance of the computer algebraic system Maple.

II. METHOD OF FINDING THE $t/x$-DEPENDENT CONSERVATION LAWS

The $n$th-order KdV equations with the general form

$$u_t = u_{nx} + H(u, u_x, \cdots, u_{(n-1)x}) \equiv F(u),$$

(1)

where $u_{ix} = \frac{\partial^i u}{\partial x^i}$, and $H$ is a specific polynomial in $u$ and its derivatives, is invariant under the following scaling transformation

$$(t, x, u) \mapsto (\mu^{-n} t, \mu^{-1} x, \mu^2 u).$$

(2)

Hence the weight [3] value of the dependent variable $u$, denoted by $\omega(u)$, is 2, which is useful for us.

Combining the method given in [1], the following proposition, proved in [1], and its corollary, allow one to obtain the $t/x$-dependent conservation laws. As usual, $D_x$ and $D_t$ are total derivatives with respect to $x$ and $t$, respectively.
Proposition If $D_x^{-1}F$, denoted by $\rho$, is a conserved density of the evolution equation
\[ u_t = F, \]
then $tD_x^{-1}F + xu$ is also one of its conserved densities.

Corollary If $J$ is a conserved flux corresponding to the conserved density $D_x^{-1}F$, then
\[ tJ - xD_x^{-1}F \]
is the associated conserved flux of $tD_x^{-1}F + xu$.

Then the basic algorithm to construct the $t/x$-dependent conservation laws (see [1] for its definition) for the $n$th-order KdV equations is fairly straightforward:

- **Step 1: Determine the general form of $\rho$.**
  Find the building blocks in $u$ and its $x$-derivatives of $\rho$ that satisfy the equation $\rho = D_x^{-1}F$. This is a crucial step. In detail, the procedure proceeds as follows:
  
  1. Determine the rank $[1, 3], R$, of $\rho$. Assuming that $\rho = D_x^{-1}F$ is a conserved density of Eq. (1), then we have
     \[ F = D_x \rho. \] 
     From Eq. (3) along with Eq. (1), we know that the highest derivative of $\rho$ is $n - 1$ and $\rho$ should contain the term of $u_{(n-1)x}$. Since $\omega(\partial/\partial x) = 1$ [1, 3], $R$ is then obtained as
     \[ R = (n - 1)\omega(\partial/\partial x) + \omega(u) = n + 1. \]
  2. Form the basis set $B$. It consists of such elements as $[M, \text{Rank}(M)]$, where $M$ is one of the monomials with rank $R$ or less, by taking all appropriate combinations of different powers of $u$; $\text{Rank}(M)$ is the rank of the monomials $M$.
  3. Form the set $Q$ that contains all monomials in $u$ and its $x$-derivatives with rank $R$. To do this, for each element in the set $B$, compute $l_i = R - \text{Rank}(M_i)$, which forms a list $L$. Then compute the $x$-derivative of $M_i$ up to $l_i$, such that the new generating monomials exactly have rank $R$. Gather the new generating monomials in the set $Q$.
  4. Linear combination of the monomials in the set $Q$ with constant coefficients $c_i$’s directly yield the general form of $\rho$. In contrast to the method of constructing the polynomial type conservation laws in [1], no terms are removed here.

Carrying on with the 3rd-order KdV equation
\[ u_t = p uu_x + u_{3x} \equiv F, \] 
where $p$ is an arbitrary constant, apparently the $\rho$ that satisfies Eq. (3) should contain the term of $u_{2x}$. Thus the rank of $\rho$ is 4, and the basis set $B = \{ [u, 2], [u^2, 4] \}$. Easily, we get the list $L = [2, 0]$. Then we compute the various $x$-derivatives of $u$ and $u^2$ up...
to 2 and 0, respectively, such that the new generating monomials exactly have rank 4. Therefore, we have
\[ \frac{d^2}{dx^2} u = u_{2x}, \quad \frac{d^3}{dx^3} (u^2) = u^2. \]

Gathering the monomials on the right hand side of the above two expressions, we derive the set \( Q = \{ u_{2x}, u^2 \} \). Hence, the general form of \( \rho \) is

\[ \rho = c_1 u^2 + c_2 u_{2x}. \]

**Step 2: Determine the unknown coefficients in \( \rho \).**
Compute \( D_x \rho \) and, in view of Eq. (3), determine the \( c_i \)'s. For example, for Eq. (4), we have \( p u u_x + u_{3x} = 2c_1 u u_x + c_2 u_{3x} \). Thus, \( \{ c_1 = p/2, c_2 = 1 \} \), and

\[ \rho = \frac{1}{2} pu^2 + u_{2x}. \]

**Step 3: Rewrite \( \rho \) as its equivalent form \( \tilde{\rho} \).**
First, rewrite \( \rho \) into the form

\[ \rho = \triangle + [\bullet]', \]

as far as one can, where \( \triangle \) is a polynomial in \( u \) and its derivatives and the symbol \( ' \) denotes the derivative of the expression \( \bullet \) with respect to \( x \). For example, the monomial \( u u_{3x} u_{5x} = (u u_{3x} u_{4x})' - (u u_{3x} u_{4x})' \) and \( u_x u_{3x} u_{4x} = (u_x u_{3x} u_{4x})' - u_{2x} u_{3x}^2 \). Thus the monomial \( u u_{3x} u_{5x} \) is finally reexpressed as \( (u u_{3x} u_{4x} - u_x u_{3x}^2)' + u_{2x} u_{3x}^2 - u u_{4x}^2 \).

For a polynomial \( \rho \) in \( u \) and its derivatives, do the same thing repeatedly. \( \rho \) can be rewritten as (6). Then, from the knowledge that the conserved densities are equivalent if they only differ by a total \( x \)-derivative we know that \( \rho \) is equivalent to

\[ \tilde{\rho} = \triangle. \]

Taking Eq. (4) as an example, Eq. (5) is equivalent to \( \tilde{\rho} = \frac{1}{2} pu^2 \) or \( u^2 \).

**Step 4: Verify whether \( \tilde{\rho} \) is a conserved density or not.**
As a matter of fact, \( u^2 \) is indeed a conserved density for Eq. (4), because Eq. (4) can be written as

\[ D_t (\tilde{\rho}) + D_x (\tilde{J}) = D_t (u^2) + D_x (-\frac{2}{3} p u^3 - 2 uu_{2x} + u_x^2) = 0. \]

By now, we verify that Eq. (5) is a conserved density of Eq. (4). For other higher order KdV equations, we can verify whether or not the equivalent form \( \tilde{\rho} \) is a conserved density by means of the procedure CONSLAW [1].
Step 5: Determine the form of the $t/x$-dependent conserved density $\bar{\rho}$.

We point out that, although in the sense of a conservation law $\rho$ is equivalent to $\bar{\rho}$, the $t/x$-dependent conserved density, denoted by $\bar{\rho}$, cannot be obtained directly by the above proposition, i.e. $\bar{\rho} \neq t\rho + xu$, instead, it should be $\bar{\rho} = t\rho + xu$. Hence, we conclude that the previously known expression, $\frac{1}{2}ptu + xu$, which is regarded as a conserved density of Eq. (4) by the authors of [2, 3], is not one. The radical reason lies in $D_x^{-1} \rho \neq F$.

Step 6: Determine the corresponding conserved flux of $\bar{\rho}$.

For Eq. (4), by using the procedure CONSLAW we find the conserved flux corresponding to $\rho = \frac{1}{2}pu^2 + xu$ to be

$$J = -\frac{1}{2}pu_x^2 - \frac{1}{3}p^2 u^3 - 2puu_{2x} - uu_x .$$

Thus, by the above proposition and the corollary, we obtain the only $t/x$-dependent conservation law for Eq. (4) as

$$D_t [t(\frac{1}{2}pu^2 + xu) + xu] + D_x [t(-\frac{1}{2}pu_x^2 - \frac{1}{3}p^2 u^3 - 2puu_{2x} - uu_x) - x(\frac{1}{2}pu^2 + uu_x)] = 0 ,$$

which is new, and is reported here for the first time.

III. $t/x$-Dependent Conservation Law for the Generalized 5th-Order KdV Equation

The generalized 5th-order KdV equation

$$u_t = au^2u_x + bu_xu_{2x} + cuu_{3x} + uu_5x \equiv F(u) ,$$

(8)

with constant parameters $a, b, c$, and $u_{kx} = \frac{\partial^k u}{\partial x^k}$, includes four well known special cases:

for $(a, b, c) = (10, 20, 30), (20, 40, 120), (30, 60, 270)\) it is the Lax equation [4]; for $(a, b, c) = (-15, -15, 45), (5, 5, 5), (30, 30, 30)\) it is an equation, called the SK equation, due to Sawata and Kotera [5], and also the case $(5, 5, 5)$ due to Dodd and Gibbon [6]; for $(a, b, c) = (10, 25, 20), (30, 75, 180)\) it is an equation, called KK due to Kaup [7] and Kupershmidt; for $(a, b, c) = (3, 6, 2)\), it is an equation, called Ito, due to Ito [8].

By a simple transformation $u = u/a$, Eq. (8) is reduced to

$$u_t + uu_{3x} + \alpha uu_xu_{2x} + \beta u^2u_x + uu_5x = 0 ,$$

(9)

where $\alpha = b/a$ and $\beta = c/a^2$. Especially, the Lax equation is obtained from Eq. (9) for $\alpha = 2, \beta = \frac{3}{10}$. The Sawada, KK, and Ito equations correspond to $(\alpha, \beta) = (1, \frac{1}{5}), (\frac{5}{2}, \frac{1}{3}), (2, \frac{5}{9})$ respectively.
Through the same analysis as used above we find the general form of \( \rho \) that satisfies Eq. (3) to be

\[
\rho = c_1 u^3 + c_2 u_x^2 + c_3 u_{4x} + c_4 uu_{2x}.
\]

In view of \( D_x^{-1}F = \rho \), i.e. \( F = D_x \rho \), we obtain

\[
\rho = \frac{\beta}{3} u^3 + \frac{\alpha - 1}{2} u_x^2 + u_{4x} + uu_{2x},
\]

which can also be written as

\[
\rho = \frac{\beta}{3} u^3 + \frac{\alpha - 3}{2} u_x^2 + (u_{3x} + uu_x)',
\]

and is equivalent to

\[
\tilde{\rho} = \frac{\beta}{3} u^3 + \frac{\alpha - 3}{2} u_x^2, \quad \text{or} \quad \tilde{\rho} = \frac{2\beta}{3(\alpha - 3)} u^3 + u_x^2.
\]

By using CONSLAW \([1]\) we verify that \( \tilde{\rho} \) is indeed a conserved density for Eq. (9) under the parameter constraints

\[
\{ \beta = \frac{(2\alpha - 1)(\alpha - 3)}{10}, \quad \alpha : \text{free} \}.
\]

Hence, \( \rho \) is a conserved density that satisfies the equation \( D_x^{-1}F = \rho \). Thus we get the \( t/x \)-dependent conserved density as

\[
\bar{\rho} = t\left[ \frac{(2\alpha - 1)(\alpha - 3)}{30} u^3 + \frac{\alpha - 1}{2} u_x^2 + uu_{2x} + u_{4x} \right] + xu.
\]

For brevity, the corresponding conserved flux is omitted here.

IV. \( t/x \)-DEPENDENT CONSERVATION LAW FOR THE GENERALIZED 7th-ORDER KdV EQUATION

The generalized 7th-order KdV equation,

\[
u_t = F = au^3u_x + bu_x^3 + cuuu_xu_{2x} + du^2u_{3x} + eu_{2x}u_{3x} + fu_{4x}u_{5x} + uu_{5x} + uu_{7x}, \quad (10)
\]

with six constant parameters \( a, b, c, d, e, f \), and \( u_{kx} = \partial^k/\partial x^k \) includes three cases that are of particular interest: (i) \( (a,b,c,d,e,f) = (\frac{4}{14}, \frac{5}{14}, \frac{10}{14}, \frac{7}{14}, 5, 3) \) (Lax equation) \([8]\); (ii) \( (a,b,c,d,e,f) = (\frac{4}{14}, \frac{1}{7}, \frac{6}{7}, \frac{7}{2}, 3, 2) \) (SK-Ito equation) \([8]\); (iii) \( (a,b,c,d,e,f) = (\frac{4}{11}, \frac{5}{11}, \frac{2}{11}, \frac{3}{11}, \frac{6}{11}, \frac{7}{11}) \) (found in \([1]\)).

By analysis, we find that the rank of \( \rho \) that satisfies \( D_x^{-1}F = \rho \) is 8. The general form of \( \rho \) is

\[
\rho = c_1 uu_x^2 + c_2 u_{2x}^2 + c_3 uu_{3x} + c_4 u_x^2u_{2x} + c_5 uu_{4x} + c_6 u^4 + c_7 u_{6x}.
\]
Thus, in view of the equation \( F = D_x \rho \), we get

\[
\rho = bu_x^2 + \frac{e - f + 1}{2}u_{2x} + (f - 1)u_x u_{3x} + \frac{c - 2b}{2}u^2 u_{2x} + uu_{4x} + \frac{a}{4}u^4 + u_{6x}, \tag{12}
\]

which can also be written as

\[
\rho = (3b - c)u_{2x}^2 + \frac{e - 3f + 5}{2}u_{2x}^2 + \frac{a}{4}u^4 + [(f - 2)uu_{2x} + \frac{c - 2b}{2}u^2 u_x + u_{5x}'], \tag{13}
\]

and is equivalent to

\[
\dot{\rho} = (3b - c)u_{2x}^2 + \frac{e - 3f + 5}{2}u_{2x}^2 + \frac{a}{4}u^4. \tag{14}
\]

By using CONSLAW we verify that the \( \dot{\rho} \) that satisfies the equation \( D_x^{-1}F = \rho \) is a conserved density of Eq. (10) under the parameter constraints

\[
\{a = \frac{(4f - f^2 - 15f + 45f + 49c - 20)}{882}, b = \frac{-9f + 2f^2 + 4 + 14c}{42}, c = \frac{7c - 4 + 5c - 2f^2}{42}, e = 2f - 1\}.
\]

Thus, by the above proposition, the \( t/x \)-dependent conservation law is

\[
\dot{\rho} = \{ \frac{(2f - 1)(f - 1)}{14}uu_{2x}^2 + \frac{4 - f}{2}u_{2x}^2 + \frac{(4 - f)(10f^2 - 45f + 49c + 20)}{3028}u^4 + [(f - 2)uu_{2x} + \frac{2e + 9f - 2f^2 - 4}{42}u^2 u_x + u_{5x}'] \} + xu.
\]

The corresponding conserved flux is also omitted here.

**V. \( t/x \)-DEPENDENT CONSERVATION LAW FOR THE 9th-ORDER KdV EQUATION**

For the 9th-order KdV equation, we consider its usual form

\[
u_t = 35uu_{2x}u_{3x} + \frac{21}{6}u_x u_{2x}^2 + 21uu_x u_{4x} + \frac{35}{6}uu_{2x}^3 + \frac{35}{3}u^2 u_x u_{2x} + 42u_{3x} u_{4x} + 28u_{2x} u_{5x}
+12u_x u_{6x} + 3uu_{7x} + \frac{161}{6} u_x^2 u_{3x} + \frac{7}{2}u^2 u_{5x} + \frac{23}{16}u^3 u_{3x} + \frac{35}{12}u^4 u_x + u_{9x} \equiv F(u).
\]

To obtain its \( t/x \)-dependent conservation law, we first compute the polynomial type conserved density of order 8, denoted by \( \rho \), which depends only on \( u \) and its derivatives. Since Eq. (15) has the same scaling property as Eq. (1), the rank of \( \rho \) is 10. Hence, according to the above algorithm, we find the general form of \( \rho \) that satisfies the equation \( D_x^{-1}F = \rho \) to be

\[
\rho = c_1u^2 u_x^2 + c_2u_{3x}^2 + c_3u_{2x} u_{4x} + c_4u^5 + c_5u_x u_{5x} + c_6uu_{6x} + c_7u^2 u_{2x} + c_8uu_{2x}^2 + c_9uu_x u_{3x}
+ c_{10}u^2 u_{4x} + c_{11}u^3 u_{2x} + c_{12}u_{8x}.\]
In view of the equation $D_x F = \rho$, we can determine the unknown $c_i$'s and then get
\[
\rho = \frac{35}{12} u^2 u_x^2 + \frac{23}{2} u^3_x + 19 u u_{2x} u_{4x} + \frac{7}{12} u^5 + 9 u_x u_{5x} + 3 u u_{6x} + \frac{7}{6} u_x^2 u_{2x} + \frac{21}{2} u u_{2x}^2 \\
+ 14 u u_x u_{3x} + \frac{7}{2} u^2 u_{4x} + \frac{35}{18} u^3 u_{2x} + u_{8x},
\]
which can be written as
\[
\rho = \frac{7}{72} u^5 - \frac{35}{12} u^2 u_x^2 - \frac{3}{2} u^3_{3x} + \frac{7}{2} u u_x^2 + [6 u u_x u_{4x} + 13 u_{2x} u_{3x} + 3 u u_{5x} + \frac{35}{18} u_x^3 + 7 u u_x u_{2x} \\
+ \frac{7}{2} u^2 u_{3x} + \frac{35}{18} u^3 u_x + u_{7x}],
\]
and is equivalent to
\[
\tilde{\rho} = \frac{7}{72} u^5 - \frac{35}{12} u^2 u_x^2 - \frac{3}{2} u^3_{3x} + \frac{7}{2} u u_x^2 \quad \text{or} \quad \hat{\rho} = u^5 - 30 u^2 u_x^2 - \frac{108}{7} u^3_{3x} + 36 u u_x^2.
\]
As before, using the procedure CONSLAW, we can easily verify that $\tilde{\rho}$ is indeed a conserved density for Eq. (15). This shows that (16), which satisfies the equation $D_x F = \rho$, is also a conserved density of Eq. (15). Hence, the 8th-order $t/x$-dependent conserved density $\tilde{\rho}$ for Eq. (15) is generated from Eq. (16) or (17) along with the expression $t \rho + xu$.

VI. CONCLUSION

In summary, combing the approach described in [1] and the direct algebraic method presented above, we obtain the $(n - 1)$th-order $t/x$-dependent conservation laws for the generalized $n$th-order KdV equations. As a matter of fact the method is applicable for any nonlinear evolution equation $u_t = F(u, u_x, u_{2x}, \cdots)$, as long as we can find a conserved density $\rho$ that satisfies the equation $D_x^{-1}F = \rho$.

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