Approximate Symmetries and Infinite Series Symmetry Reduction Solutions to Perturbed Kuramoto–Sivashinsky Equation

YAO Ruo-Xia,1,2,3† JIAO Xiao-Yu,1 and LOU Sen-Yue1,3

1Department of Physics, Shanghai Jiao Tong University, Shanghai 200062, China
2School of Computer Science, Shaanxi Normal University, Xi’an 710062, China
3Department of Physics, Ningbo University, Ningbo 315211, China

(Received July 7, 2008)

Abstract Starting from Lie symmetry theory and combining with the approximate symmetry method, and using the package LieSYMGRP proposed by us, we restudy the perturbed Kuramoto–Sivashinsky (KS) equation. The approximate symmetry reduction and the infinite series symmetry reduction solutions of the perturbed KS equation are constructed. Specially, if selecting the tanh-type travelling wave solution as initial approximate, we not only obtain the general formula of the physical approximate similarity solutions, but also obtain several new explicit solutions of the given equation, which are first reported here.

PACS numbers: 02.30.Jr, 11.30.j

Key words: perturbed Kuramoto–Sivashinsky equation, approximate symmetry reduction, series reduction solution

1 Introduction

Consider the following partial differential equation (PDE),

\[ H + a H_{xx} + b H_{xxxxx} + \beta H + \alpha H_{x} + \varepsilon H_{x}H_{xx} = 0 \] (1)

with constant parameters \( a, b, \alpha, \) and \( \varepsilon \). It is useful to rewrite Eq. (1) in the following form after the transformation \( H = u_{x} \),

\[ u_{t} + au_{xx} + bu_{xxxxx} + \beta u + \frac{\alpha}{2} u_{x}^{2} + \frac{\varepsilon}{2} u_{xx}^{2} = 0 . \] (2)

We notice that when \( \beta = \varepsilon = 0 \), it is essentially the usual Kuramoto–Sivashinsky (KS) equation (without the stabilizing term, \( \beta = 0 \)), which modes a pattern formation in different contexts and is a paradigm of low-dimensional behavior in solutions to partial differential equations (PDEs). Kuramoto[3] derived it in the context of reaction-diffusion equations modelling the Belousov–Zabotinskii reaction. Sivashinsky[2] derived it to model small thermal diffusive instabilities in laminar flame fronts. The equation arises also, among other situations, in Poiseuille flow of a film layer on an inclined plane,[3] in solidification at large super-cooling,[4] and in the step-flow growth.[5] Numerous investigations were devoted to the KS equation, most of them focused on the transition to chaotic solutions,[6] which are exhibited by the KS equation at moderate aspect ratios. (The aspect ratio is defined as the ratio of the size of the system to the typical stability length, which corresponds to the neutral wavelength for infinitesimal perturbations about the solution \( h = 0 \).

The KS equation is one of the most prominent and generic equations that arise in nonequilibrium systems, such as hydrodynamics and moving interfaces. For a large box size the KS equation is known to exhibit spatiotemporal chaos. The KS equation with a linear stabilizing term (the so-called stabilized KS equation) occurs in many situations: (i) directional solidification where kinetics are decisive and (ii) terrace-edge evolution in step-flow growth in the presence of step-step interaction. Authors Chaouqui Misbah and Alexandre Valance[7] presented an extensive analytical and numerical study of the stabilized KS equation. It is found that this equation reveals a variety of secondary bifurcations, and hence is very important.

In this paper, we will take account of the following equation,

\[ u_{t} + au_{xx} + bu_{xxxxx} + \alpha u_{x}u_{x} + \varepsilon u_{x}u_{xx} = 0 , \quad |\varepsilon| \ll 1 , \] (3)

which belongs to the perturbed KS equation with the weak dissipation term \( \varepsilon u_{x}u_{xx} \). Equation (3) can be treated by means of the approximate symmetry perturbation approach, which is an integration of perturbation method and the symmetry reduction method first proposed by Fushchich,[6] with the name approximate symmetry. The crucial point is that a physical small parameter appears when equations are cast into a dimensionless form and a perturbation solution is possible in terms of this small parameter.

This paper begins with an expanding of infinity series of \( u \) about a small parameter \( \varepsilon \). After the substitution of \( u \), one can change the given PDE (3) into a system of PDEs. Then, we manage to reduce the newly obtained system of PDEs by considering its Lie point symmetries and constructing the corresponding invariant variables. It is lucky that we reduce the system of PDEs into a system of ODEs successfully. Next, it is reasonable for us

†The project supported by National Natural Science Foundations of China under Grant Nos. 10775030, 10475055, and 90503006, the Natural Science Research Plan in Shaanxi Province under Grant No. 508409, the Research Fund of Postdoctoral of China under Grant No. 2007010727, and the Research Found of Shaanxi Normal University

‡Corresponding author, E-mail: rayao@sjtu.edu.cn
to try to search for its exact analytical solutions from the viewpoint of physics. It is also very lucky that we can obtain its all solitary wave solutions step by step. Section 2 is devoted to the application of the approximate symmetry perturbation approach to Eq. (3). Section 3 aims at constructing the general travelling wave reduction and the solitary wave solutions of Eq. (3) via the approximate symmetry perturbation approach. The last section is a conclusion and discussion of the results.

2 Approximate Symmetry Reduction Approach

According to the perturbation theory, in the actual case, solutions of perturbed partial differential equations can be expressed as the form containing finite portion of the series up to some order of a small parameter. Obviously, one can consider higher orders of approximation of \( u \) in \( \varepsilon \), i.e., \( u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \), and can study the symmetry of the corresponding approximate system of the PDE for functions \( u_j, j = 0, 1, 2, \ldots \). Now, we start directly from the following infinity case. For Eq. (3), the solution can be supposed to be of the general form

\[
u = \sum_{j=0}^{\infty} \varepsilon^j u_j,
\]

where \( u_j \) are functions of \( x \) and \( t \).

Substituting Eq. (4) into Eq. (3) and vanishing the coefficients of various powers of \( \varepsilon \), we obtain a system of partial differential equations denoted as \( \mathcal{F} \),

\[
O(\varepsilon^0) : \quad u_{0,t} + b u_{0,xxxx} + a u_{0,xx} + \alpha u_{0,x} = 0,
O(\varepsilon^1) : \quad u_{1,t} + b u_{1,xxxx} + a u_{1,xx} + \left( \alpha u_0 u_1 + \frac{1}{2} u_0^2 \right)_x = 0,
O(\varepsilon^2) : \quad u_{2,t} + b u_{2,xxxx} + a u_{2,xx} + \left( \alpha (u_0 u_2 + \frac{1}{2} u_1^2) + u_0 u_{1,x} \right)_x = 0,
O(\varepsilon^3) : \quad u_{3,t} + b u_{3,xxxx} + a u_{3,xx} + \left( \alpha (u_0 u_3 + u_1 u_2) + u_0 u_{x,xx} + \frac{1}{2} u_1^2 \right)_x = 0,
O(\varepsilon^4) : \quad u_{j,t} + b u_{j,xxxx} + a u_{j-1,xx} + \frac{1}{2} \sum_{k=0}^{j} (\alpha u_k u_{j-k} + u_{k,xx} u_{j-k-1,x}) = 0,
\]

with \( u_{-1} = 0 \).

To find some exact solutions of the above system, we first construct its Lie point symmetries and then give the corresponding symmetry reductions. Supposing the Lie point symmetry of Eqs. (5) is of the form

\[
\sigma_j = X \frac{\partial}{\partial x} u_j + T \frac{\partial}{\partial t} u_j - U_j, \quad (j = 0, 1, \ldots),
\]

where \( X, T \), and \( U_j (j = 0, 1, \ldots) \) are functions with respect to \( x, t \), and \( u_j(j = 0, 1, \ldots) \), which means that the system of Eqs. (5) is invariant under the following transformations,

\[
x \rightarrow x + \varepsilon \sigma_j, \quad t \rightarrow t, \quad u_j \rightarrow u_j + \varepsilon \sigma_j + O(\varepsilon^2)
\]

with a small parameter \( \varepsilon \). Obviously, \( \sigma_j(j = 0, 1, \ldots) \) satisfies

\[
\frac{\partial}{\partial \varepsilon} \mathcal{F}(u_j + \varepsilon \sigma_j) \bigg|_{\varepsilon = 0} = 0.
\]

Then from condition (8) and Eqs. (5) we get the linearized equations of Eqs. (5) listed bellow,

\[
\sigma_{0,1} + b \sigma_{0,xxxx} + a \sigma_{0,xx} + \alpha (u_0 \sigma_0)_x = 0,
\sigma_{1,1} + b \sigma_{1,xxxx} + a \sigma_{1,xx} + \alpha (u_0 \sigma_1 + u_1 \sigma_0)_x = 0,
\sigma_{2,1} + b \sigma_{2,xxxx} + a \sigma_{2,xx} + \alpha (u_0 \sigma_2 + u_1 \sigma_1 + u_2 \sigma_0)_x = 0,
\sigma_{3,1} + b \sigma_{3,xxxx} + a \sigma_{3,xx} + \alpha (u_0 \sigma_3 + u_1 \sigma_2 + u_2 \sigma_1 + u_3 \sigma_0)_x = 0,
\sigma_{j,1} + b \sigma_{j,xxxx} + a \sigma_{j,xx} + \frac{1}{2} \sum_{k=0}^{j} (\alpha (u_k \sigma_{j-k} + u_{k,xx} u_{j-k-1}) = 0
\]

with \( u_{-1} = \sigma_{-1} = 0 \).

3 Symmetry Perturbation of Travelling Wave Solutions

When \( t_0 \neq 0 \), from Eq. (10), we can choose the similarity variable as

\[
\xi = x - \frac{\alpha c_1 l^2}{2l_0} - \frac{x_0}{l_0} t.
\]

Then the similarity solutions for the fields \( u_j \) are

\[
u_0 = \frac{x_0}{l_0} + c_1 l^2 V_0(\xi), \quad u_1 = V_1(\xi),
\]

\[
u_2 = V_2(\xi), \ldots, \quad u_j = V_j(\xi), \ldots
\]

Accordingly, the perturbation series solution of Eq. (3) is of the form

\[
u = \frac{x_0}{l_0} + c_1 l^2 + \sum_{j=0}^{\infty} \varepsilon^j V_j(\xi),
\]
and the similarity reduction equations related to the similarity solutions (12) are

$$O(e^0) : \ aV_0,_{\xi\xi} + bV_0,_{\xi\xi\xi\xi} + aV_0V_0,_{\xi} + \frac{c_1}{l_0} = 0,$$

$$O(e^1) : \ aV_1,_{\xi\xi} + bV_1,_{\xi\xi\xi\xi} + \left(\frac{1}{2}V_0^2,_{\xi} + aV_0V_1\right),_{\xi} = 0,$$

$$O(e^2) : \ aV_2,_{\xi\xi} + bV_2,_{\xi\xi\xi\xi} + \left[V_0V_1,_{\xi\xi} + a\left(\frac{1}{2}V_1^2,_{\xi} + V_0V_2\right)\right],_{\xi} = 0,$$

$$O(e^3) : \ aV_3,_{\xi\xi} + bV_3,_{\xi\xi\xi\xi} + \left[V_0V_2,_{\xi\xi} + \frac{1}{2}V_2^2,_{\xi\xi} + a(V_0V_3 + V_1V_2)\right],_{\xi} = 0,$$

$$O(e^j) : \ aV_j,_{\xi\xi} + bV_j,_{\xi\xi\xi\xi} + \sum_{k=0}^{j}(V_{k,\xi}V_{j-k-1,\xi} + aV_{k}V_{j-k,\xi}) = 0,$$

With $V_{-1} = 0$.

As a matter of fact, the above results all can be obtained using the package LieSYMGRP prosed by us.\cite{9}

When $c_1 = 0$, solving Eq. (14a) we can get the following tanh profile solution with the form

$$V_0,_{\xi} = \sum_{i=0}^{3} a_i \tanh^i(k\xi), \quad (15)$$

where

$$a_3 = \frac{12b}{k}, \quad a_1 = \frac{360b}{k}, \quad a_0 = 0, \quad a_2 = 0, \quad a = -76bk^2, \quad (16)$$

Then substituting Eq. (15) along with Eqs. (16) and (17) into Eq. (14b) respectively, we get

$$V_1,_{\xi} = \sum_{i=0}^{5} b_i \tanh^i(k\xi), \quad (18)$$

where

$$a = -76bk^2, \quad b_4 = b_0, \quad b_2 = -2b_0, \quad b_1 = \frac{720b^5k}{6}, \quad b_3 = \frac{-1440b^5k}{9}, \quad b_2 = \frac{720b^5k}{18}, \quad (19)$$

The following tanh profile solution with the form

$$V_2,_{\xi} = \sum_{i=0}^{9} a_i \tanh^i(k\xi), \quad (20)$$

where

$$a_3 = \frac{12b}{k}, \quad a_1 = \frac{360b}{k}, \quad a_2 = \frac{1080b^3k}{2}, \quad a_0 = 0, \quad a_2 = 0, \quad a = 11b, \quad (21)$$

From now on, other solutions corresponding to the case of Eq. (17) are no longer given.

Solution $V_3(k\xi)$ is

$$V_3,_{\xi} = \sum_{i=0}^{9} a_i \tanh^i(k\xi), \quad (22)$$

\[ e_2 = \frac{388800k^8b^2d_2a + 194400b_0k^10b^2 + b_0^3k^4 - 77760b_0k^6c_0^2}{38880^2k^6a^2}, \]

\[ e_5 = \frac{507546000k^12b^2 - 957b_0^2k^2\alpha^4 + 58 a^3b_0d_0}{5220b^2k^2\alpha^4}, \]

\[ e_6 = \frac{388800k^8b^2d_2a - 8067600b_0k^10b^2 + b_0^3k^4}{38880^2k^6a^2}, \]
where

\[ \frac{-5637600k^{10}b^{2}d_{0}\alpha + 279255600k^{12}b^{2}b_{0} - 232\alpha^{4}b_{0}^{2}k^{2} + 87\alpha^{5}b_{0}^{2}d_{0}}{1127520b^{2}k^{6}\alpha^{3}} + \frac{1127520b^{2}k^{6}\alpha^{3}}{1127520b^{2}k^{6}\alpha^{3}} \]


As much as we know, these solutions are significant and first reported here.

4 Conclusion Remarks

In summary, the most prominent and generic perturbed KS equation is studied by applying the approximate symmetry perturbation approach and using the package LieSYMGRP. The approximate symmetry reductions and the infinity series symmetry reduction solutions of the perturbed KS equation are constructed. Also, if choosing tanh-type travelling wave solution as initial approximate, luckily we obtain the general formula of the physical infinite series symmetry reduction solutions for the perturbed KS equation.