Conservation laws and new exact solutions for the generalized seventh order KdV equation

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Abstract

Abundant polynomial type conservation laws are constructed for a seventh order nonlinear evolution equation with six arbitrary parameters. From the parameters constraints that leading to the existence of conserved densities, a new unnamed seventh order KdV type equation that may be integrable is reported. By introducing nonlinear transformation, the new soliton solution as well as the solitary wave solution to the new unnamed seventh order KdV type equation are obtained.

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1. Introduction

In the study of system of differential equations, the concept of a conservation law, which is a mathematical formulation of the familiar physical laws of conservation of energy, conservation of momentum and so on, plays an important role in the analysis of basic properties of the solutions. The investigation of conservation laws (CLaws) of the Korteweg de Vries was the starting point of the discovery of a number of techniques to solve evolution equations [2] (Miura transformation, Lax pair, inverse scattering technique, bi-Hamilton structures). As expressed through Noether’s theorem, there is a close connection between CLaws and the Lie point symmetries of the differential equation [3,4] and the existence of a large number of CLaws of a PDE (system) is a strong indication of its integrability. Researchers have done much work on how to automatically construct the CLaws for differential equations [5,6]. May be some lower order CLaws can be obtained directly by hand, but the higher order ones, which need much more tedious work, do not seem straightforward to find, if they exist at all. The purpose of this paper is to study the properties of CLaws for the generalized seventh order KdV equation. To searching the CLaws for the GSO-KdV equation, we start from the dilation property of the GSO-KdV equation and base on a theorem given in [7], which basic idea is that the Euler operator acting on an expression gives identically zero, i.e. the Euler–Lagrange equation must vanish identically, if and only if the expression is a divergence. Then, with the aid of Maple, we obtain rich higher order polynomial type CLaws for the GSO-KdV equation. From the reported general results, we obtain the corresponding polynomial CLaws for the standard GSO-KdV (Lax), and the seventh order SK-Ito equations respectively. Furthermore, apart from covering the well known seventh order Lax and the seventh order SK-Ito equations in constructing the CLaws for the GSO-KdV equation, a new unnamed seventh order KdV (UNSO-KdV) type equation is found, which, to our knowledge, has not been studied with regards to the CLaws and the solitary wave solution or soliton solution. Here, we also obtain abundant polynomial type CLaws for the UNSO-KdV type equation. Armed with the knowledge that an evolution
equation is possible integrable if it has infinitely many or a large number of CLaws, we conjure that the UNSO-KdV type equation may be integrable. Upon that, by introducing useful nonlinear transformation, we obtain the new solitary wave and soliton solutions for the UNSO-KdV type equation.

2. Conservation laws for the GSO-KdV equation

The generalized seventh order KdV equation

\[ u_t + \tilde{a}u^4u_x + \tilde{b}u^3 + \tilde{c}u_t u_{2\xi} + \tilde{d}u^{2}u_{3\xi} + \tilde{e}u_{2\xi}u_{3\xi} + \tilde{f}u_{4\xi} + \tilde{g}u_{5\xi} + u_{7\xi} = 0, \]  

(2.1)

with seven constant parameters \( \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{f}, \tilde{g} \), and \( u_{7\xi} = \partial^7 / \partial x^7 \) includes two well known special cases \[8\]: for \( (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{f}, \tilde{g}) = (140, 70, 280, 70, 70, 42, 14) \), the Lax equation; for \( (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{f}, \tilde{g}) = (252, 63, 378, 126, 63, 42, 21) \), an equation, named SK-Ito equation.

Now, by introducing a simple transformation \( u = u / \tilde{g} \), the generalized seven parameters KdV equation is reduced to a six parameters one

\[ u_t + au^4u_x + bu^3 + cu_t u_{2\xi} + du^{2}u_{3\xi} + eu_{2\xi}u_{3\xi} + fu_{4\xi} + uu_{5\xi} + u_{7\xi} = 0, \]  

(2.2)

where \( a = \tilde{a} / \tilde{g}^3 \), \( b = \tilde{b} / \tilde{g}^2 \), \( c = \tilde{c} / \tilde{g}^2 \), \( d = \tilde{d} / \tilde{g}^2 \), \( e = \tilde{e} / \tilde{g} \), \( f = \tilde{f} / \tilde{g} \). Then the Lax equation is obtained for \( (a, b, c, d, e, f) = (5/98, 5/14, 10/7, 5/14, 5/3) \), and the SK-Ito equation is obtained for \( (a, b, c, d, e, f) = (4/147, 1/7, 6/7, 2/7, 3/2) \).

Now, we study the polynomial type CLaws for Eq. (2.2), which are of interest for its exact solutions, for its understanding, classification and for supporting its numerical solutions.

Eq. (2.2) is invariant under the scaling symmetry

\[(t, x, u) \rightarrow (\lambda^{-1} t, \lambda^{-1} x, \lambda^2 u), \]

(2.3)

where \( \lambda \) is an arbitrary parameter. Obviously, \( u \) corresponds to two derivatives with respect to (w.r.t) \( x \), and \( \partial / \partial t \) corresponds to seven derivatives w.r.t \( x \), which can be alternatively expressed as \( u \sim \partial^2 / \partial x^2 \) and \( \partial / \partial t \sim \partial^7 / \partial x^7 \). That motivates us that we can assign such values of weight, denoted by \( \omega \), to the dependent variables and differentials in Eq. (2.2). The weight is assumed to be nonnegative or rational. Generally, we can set \( \omega(\partial / \partial x^2) = i, i = 1, 2, \ldots \), then for Eq. (2.2), we have

\[ \omega(u) = 2, \quad \omega(\partial / \partial t) = 7. \]

The total weight of each factor of a monomial, \( M \), is named the rank of that monomial, denoted by \( \text{Rank}(M) \). It is easy to see that all monomials in Eq. (2.2) have the same rank 7. This property is called uniformity in rank, i.e. the GSO-KdV equation is consistent in rank.

It is well known that the seventh order Lax equation, ranked in the hierarchy of KdV, has infinitely many conservation laws. The first two are

\[ (u)_t + \left( u_{4\xi} + \frac{5}{392} u^4 + \frac{5}{14} uu_{2\xi} + \frac{5}{14} u^2 u_{3\xi} + \frac{3}{2} u_{2\xi} + 2u_{3\xi} + uu_{4\xi} \right)_x = 0, \]  

(2.4)

\[ (u^2)_t + \left( \frac{1}{49} u^5 + \frac{5}{14} u^2 u_{2\xi} + \frac{5}{7} uu_{2\xi} + 2u_{4\xi} + uu_{4\xi} + 2u_{2\xi}u_{4\xi} + 2u_{4\xi} + 2u_{6\xi} = 0, \]  

(2.5)

represent the conservation of momentum and energy respectively. The ranks of Eqs. (2.4) and (2.5) and are 9 and 11 respectively. We see that the property of uniformity in rank does not change. Obviously, the expressions in the brackets of Eqs. (2.4) and (2.5) are polynomials in \( u \) and \( u_{\xi} \), and do not depend explicitly on \( x \) and \( t \). Such CLaws are called polynomial type CLaws. Generally, a polynomial type CLaw that is to be fulfilled by solutions of Eq. (2.2) is

\[ \rho_t + J_x = 0, \]  

(2.6)

where \( \rho \), the conserved density, and \( J \), the associated conserved flux.

Once the weight of the dependent variable \( u \) in Eq. (2.2) in hand, one can carry through the following steps to construct the polynomial type CLaws for Eq. (2.2). We mention that the essence of calculating the CLaws is based on the Theorem 4.7 \[7, p. 248\].
• Determining the form of the conserved density \( \rho \), i.e. finding the building blocks in \( u \) and their \( x \)-derivatives of \( \rho \) with prescribed rank \( R \). Note that the prescribed \( R \) should be a multiple of \( \omega(u) = 2 \).

For example, we compute the form of the density of rank \( R = 6 \). The procedure proceeds as follows:

1. Building a basis set \( B \). It consists of such elements as \([ M_i, \text{rank}(M_i)]\), where \( M_i \) is one of the monomials of rank \( R \) or less by taking all appropriate combinations of different powers of the elements in set \( V \), and \( \text{rank}(M_i) \) is the rank of the monomials \( M_i \). Here, \( B = \{[u, 2], [u^2, 4], [u^3, 6]\} \).

2. Computing \( l_i = R - \text{rank}(M_i) \), \( (i = 1, 2, 3) \) and collecting them in a list \( L = [4, 2, 0] \).

3. Computing the various \( x \)-derivatives of \( u \), \( u^2 \) and \( u^3 \) up to 4, 2 and 0 respectively such that the new generating monomials exactly have rank 6. Therefore, we have

\[
\frac{d^4}{dx^4} u = u_{4s}, \quad \frac{d^2}{dx^2} (u^3) = 2uu_{2s} + 2u^2_s, \quad \frac{d^0}{dx^0} (u^3) = u^3.
\]

(2.7)

Gathering all the monomials in the right hand sides of the equations in (2.7), we derive a set \( Q = \{u_{4s}, uu_{2s}, u^2_s, u^3\} \).

4. Removing the redundant monomials in set \( Q \). It is a key step. It means that, the terms that can be written as a derivative w.r.t. \( x \), or as a derivative up to terms kept (prior) in set \( Q \) should be removed, which means that they belong to a same equivalent class. For example, for a monomial \( M = u_{4s}u_{2s}, \) if \( \text{mod}(m + n, 2) = 0 \) then \( M \) should be replaced by \( u^2_s \) for the reason of \( u_{4s}u_{2s} = [u^{(k-1)}_{4s}]u^{(k)}_{2s} = u^2_s, k = \frac{m + n}{2} \); if \( \text{mod}(m + n, 2) \neq 0 \) then \( M \) should be removed for the reason of \( u_{4s}u_{2s} = (u^2_s)^k, k = \frac{m + n}{2} \). Therefore, the monomial \( uu_{2s} \) is replaced by \( u^2_s \), and \( u_{4s} \) is cancelled. Thus, we obtain the simplified set \( P = \{u^2_s, u^3\} \).

5. Linearizing combination of the elements in set \( P \) with constant coefficients \( c_i \)'s yields the form of the polynomial conserved density of rank 6 as

\[
\rho = c_1u^2_s + c_2u^3.
\]

(2.8)

• Determining the unknown coefficients \( c_1 \) and \( c_2 \). Computing the \( t \)-derivative of \( \rho \) and eliminating all \( t \)-derivatives of \( u \) by using Eq. (2.2) give an expression \( P \), which must be a total \( x \)-derivative. Based on the theorem 4.7 [7], we know that the Euler–Lagrange equation must vanish identically, i.e. \( E(P) = 0 \), \( E \) is Euler operator defined in [7, p. 366].

• Grouping the remained terms that including the dependent variable \( u \) and the different \( x \)-derivatives of \( u \) and setting them be zero. This yields a linear system for \( c_1 \) and \( c_2 \), which on solving gives

\[
\{c_2 = -7c_1, c_1 = c_1\}.
\]

For free \( c_1 \), without loss of generality, we choose it to be 1. Then, the conserved density of rank 6 for the seventh order Lax equation is

\[
\rho = u^3 - 7u^5.
\]

(2.9)

• Determining the corresponding conserved flux of rank 6. Once the conserved density \( \rho \) determined, then, the corresponding conserved flux \( J \) can be got from Eq. (2.6) by using the idea of integrating by parts. The associated conserved flux of rank 6 is

\[
J = -10uu_{4s}^2 + \frac{5}{14}uu^3u^2 - 36uu_{4s}^2u_{2s} - 20 uu_{2s}^2u_{2s} + 10 uu_{2s}^2u_{4s} + 3 uu^2_{2s} + \frac{15}{14} uu^4_{2s} + \frac{5}{196} uu^6 - 14 uu_{4s} + \frac{15}{4} uu^4_{4s} - 5 uu^2_{4s}u_{3s} \\
- 20 uu_{4s} + 3 uu_{4s}^3 + 14 uu_{2s}u_{2s} + 20 uu_{2s}u_{4s} - 60 uu_{2s} + 20 uu_{2s}^3 - 14 uu_{4s} + 7 uu^2_{4s},
\]

which represents the conservation of the Hamiltonian.

We write a Maple procedure, named CONSLAW, to search more polynomial type CLaws for Eq. (2.2), which automates the above described computations and delivers, if any, the possible parameters constraints and the associated conservation laws for a prescribed rank \( R \) of \( \rho \). It is notable that when constructing the polynomial CLaws of a fixed rank \( R \), the procedure CONSLAW first gives all possible branches of parameters constraints so that a possible sequence of conservation laws exist, irrespective whether or not those branches could finally lead to a conserved density. Naturally, certain parameter conditions do lead to conservation law, while some do not. For whatever situations, CONSLAW can deal with them separately, and then reports all the polynomial type CLaws for the different parameters constraints.
To do so, we proceeds
\[ > \text{CONSLAW}([\text{diff}(u(x,t),t) + a \cdot u(x,t)^3 \cdot \text{diff}(u(x,t),x) + b \cdot \text{diff}(u(x,t),x)^3 + c \cdot u(x,t) + \text{diff}(u(x,t),x) \cdot \text{diff}(u(x,t),x^3) + d \cdot \text{diff}(u(x,t),x) + e \cdot \text{diff}(u(x,t),x) + f \cdot \text{diff}(u(x,t),x) + g \cdot \text{diff}(u(x,t),x) + h \cdot \text{diff}(u(x,t),x) + i \cdot \text{diff}(u(x,t),x) + j \cdot \text{diff}(u(x,t),x) + k \cdot \text{diff}(u(x,t),x) + l \cdot \text{diff}(u(x,t),x) + m \cdot \text{diff}(u(x,t),x) + n \cdot \text{diff}(u(x,t),x) + o \cdot \text{diff}(u(x,t),x) + p \cdot \text{diff}(u(x,t),x) + q \cdot \text{diff}(u(x,t),x) + r \cdot \text{diff}(u(x,t),x) + s \cdot \text{diff}(u(x,t),x) + t \cdot \text{diff}(u(x,t),x) + u \cdot \text{diff}(u(x,t),x) + v \cdot \text{diff}(u(x,t),x) + w \cdot \text{diff}(u(x,t),x) + x \cdot \text{diff}(u(x,t),x) + y \cdot \text{diff}(u(x,t),x) + z \cdot \text{diff}(u(x,t),x) + \text{diff}(u(x,t),x^3)]); \]

Then, abundant polynomial type CLaws for various ranks of conserved densities for Eq. (2.2) is reported. Here, the first seven representative CLaws for Eq. (2.2) are listed as follows.

**Case of Rank**\((\rho) = 2\): CONSLAW delivers one set of conserved density–flux pair immediately:

\[ (u), \left( u_{0x} + \frac{a}{4} u^4 + bu_x + du_x^2 + eu_x^2 + \frac{f}{2} u_{2x}^2 + fu_{x^2} - \frac{e}{2} u_{xx}^2 + uu_x - u_x u_{xx} + \frac{1}{2} u_x^2 \right) = 0, \]  

(2.10)

for which, the parameters constraint is \(c = 2(b + d)\). Eq. (2.10) represents the conservation of momentum. By simply checking, we see that the Lax and SK-Ito equations all possess this CLaw.

**Case of Rank**\((\rho) = 4\): CONSLAW reports, within 0.97 s, the only one branch of parameters constraints as \(\{c = b + 3d, e = 5f - 10\}\) and then gives one set of conserved density–flux pair for Eq. (2.2) as

\[ (u^2), \left( \frac{2a}{5} u^5 + \frac{2c}{3} u^3 u_x + \frac{2b}{3} u^3 u_{xx} + \frac{4e}{5} uu_x + \frac{2e}{5} uu_x u_{xx} + 2u^2 u_{xx} + 2uu_x + bu_{xx} - \frac{2e}{3} uu_x u_{xx} - 2u_x u_{xx} \right) = 0, \]  

(2.11)

which represents the conservation of energy. Specially, only the Lax equation admits this conservation law.

**Case of Rank**\((\rho) = 6\): CONSLAW finds that only under the parameters constraints

\[ \left\{ \begin{array}{l} a = \frac{28bf - 42b - 720f^2 + 1350f - 24ef^2 + 72ef + 120f^3 + 126cf - 54e - 189c - 81}{294} \\ d = \frac{45 + 14c + 3e - 45f + 10f^2 - 2ef}{14} \end{array} \right\}, \]

(2.12)

Eq. (2.2) admits the following one set of conserved density

\[ \rho = u_x^3 + \frac{3 - 2f}{21} u^4, \]  

(2.13)

the corresponding conserved flux includes 75 terms omitted here. The running time is 11.878 s. By simply checking, we see that both the Lax and SK-Ito equations admit this conservation law.

**Case of Rank**\((\rho) = 8\): CONSLAW also delivers one branch of parameters constraints as

\[ \left\{ \begin{array}{l} a = \frac{196c + 80 - 200f + 85f^2 - 49cf - 10f^3}{882} \\ b = \frac{-9f + 2f^2 + 4 + 14c}{42} \\ d = \frac{7z - 4 + 9f - 2f^2}{42} \end{array} \right\} \]

(2.14)

and gives one set of conserved density–flux pairs for Eq. (2.2) within 54.814 s. The number of the terms included in the conserved flux is 114, hence, we only list the conserved density as

\[ \rho = \frac{10f^2 - 45f + 49c + 20}{1764} u^4 + \frac{1 - 2f}{7} uu_x^2 + uu_{xx}^2, \]

(2.15)

It is easy to check that the Lax and SK-Ito equations all admit this conservation law.

**Case of Rank**\((\rho) = 10\): CONSLAW delivers one branch of parameters constraints as

\[ \left\{ b = \frac{c}{4}, d = \frac{c}{4}, e = 5, f = 3 \right\} \]

(2.16)

and reports one set of conserved density–flux pair. The number of the terms included in the conserved flux omitted here is 65 and the corresponding conserved density is
\[
\rho = \frac{c}{4} u^2 u_x^2 - uu_x - \frac{a}{10} u^5 + u_x^2.
\]  
(2.17)

The running time is 23.69 s. We see that the Lax equation admits this conserved density of rank 10, while the Sk-Ito equation does not.

Case of Rank \((\rho) = 12\): CONSLAW reports three branches of parameters constraints and the corresponding conserved density–flux pairs one by one within 101.282 s. There are 74 terms in each of the three conserved fluxes. Now, we only list its conserved densities.

(A) The first branch of parameter constraints is
\[
\left\{ a = \frac{5}{98}, \ b = \frac{5}{14}, \ c = \frac{10}{7}, \ d = \frac{5}{14}, \ e = 5, \ f = 3 \right\}
\]  
(2.18)

and the associated conserved density is
\[
\rho = \frac{3}{2744} u^6 + u_x^2 - \frac{5}{28} u_x^4 + \frac{10}{7} u_x^3 - \frac{9}{7} uu_x^2 - \frac{15}{98} u_x^5 + \frac{9}{14} u_x^2 u^2_x,
\]  
(2.19)

which, obviously, is only admitted by the seventh order Lax equation.

(B) The second branch of parameters constraints is
\[
\left\{ a = \frac{4}{147}, \ b = \frac{1}{7}, \ c = \frac{6}{7}, \ d = \frac{2}{7}, \ e = 3, \ f = 2 \right\}
\]  
(2.20)

and the conserved density is
\[
\rho = \frac{4}{21609} u^6 + u_x^2 - \frac{17}{147} u_x^4 + \frac{16}{21} u_x^3 - uu_x^2 - \frac{50}{1029} u_x^5 + \frac{16}{49} u_x^2 u^2_x,
\]  
(2.21)

which is only admitted by the well known SK-Ito equation.

(C) The third branch of parameters constraints is
\[
\left\{ a = \frac{4}{147}, \ b = \frac{5}{14}, \ c = \frac{9}{7}, \ d = \frac{2}{7}, \ e = 6, \ f = \frac{7}{2} \right\}
\]  
(2.22)

and the conserved density is
\[
\rho = \frac{16}{21609} u^6 + u_x^2 - \frac{31}{294} u_x^4 + \frac{37}{21} u_x^3 - \frac{10}{7} uu_x^2 - \frac{20}{147} u_x^5 + \frac{34}{49} u_x^2 u^2_x,
\]  
(2.23)

which is only admitted by the following unnamed seventh order KdV type equation
\[
u_t + \frac{4}{147} u^3 u_x + \frac{5}{14} u^3 + \frac{9}{7} uu_x u_{2x} + \frac{2}{7} u^2 u_{3x} + 6u_{2x} u_{3x} + \frac{7}{2} u_t u_{4x} + uu_{5x} + u_{7x} = 0.
\]  
(2.24)

Eq. (2.24), as much as we know, is a new type of seventh order KdV type equation. This UNSO-KdV type equation arouses our great interest that whether or not it has infinitely many conservation laws or it has solitary wave (soliton) solution or an integrable one. With such idea, we check the conditions of ranks 2–10 of the conserved densities obtained before. The results show that it admits the CLaws of (2.10), (2.13) and (2.15). Next, we continue such computations of CLaws for the GSO-KdV equation as possible as we can.

Case of Rank \((\rho) = 14\): CONSLAW also reports three branches of parameters constraints and the corresponding conserved density–flux pairs one by one within 341.479 s. There are 113 terms in each of the three conserved fluxes. Now, we only list its conserved densities.

(A) The first branch of parameters constraints is the same as (2.18). The conserved density is
\[
\rho = \frac{55}{7} uu_x u_x^2 + \frac{55}{196} uu_x^4 - \frac{11}{7} u_x^4 u - \frac{110}{49} uu_x u^2 + \frac{99}{98} uu_x^2 + \frac{165}{2744} u_{4x}^2 = \frac{33}{134456} u^7 + \frac{33}{14} uu_x^2 + \frac{33}{98} uu_x^2 - \frac{33}{98} u_x^4 u^2_x,
\]  
(2.25)

which, obviously, is admitted by the seventh order Lax equation.
(B) The second branch of parameters constraints is the same as (2.20). The conserved density is
\[
\rho = \frac{32}{7} u_2 u_3^2 + \frac{6}{49} u u_4^4 - \frac{9}{7} u_4^2 u - \frac{163}{147} u u_3^3 + \frac{29}{49} u^2 u_4^2 + \frac{5}{343} u^4 u^2 - \frac{5}{151263} u^7 + u^2 - \frac{72}{49} u_2^2 u_4^2 - \frac{19}{147} u^3 u_2^2,
\]
which is admitted by the well known SK-Ito equation.

(C) The third branch of parameters constraints is the same as (2.22). The conserved density is
\[
\rho = \frac{19}{2} u_2 u_3^2 + \frac{41}{196} u u_4^4 - \frac{12}{7} u_4^2 u - \frac{737}{294} u_3^2 u_4^2 + \frac{53}{294} u^2 u_4^2 + \frac{15}{294} u^4 u_4^2 - \frac{20}{151263} u^7 + u^2 - \frac{291}{196} u_2^2 u_4^2 - \frac{46}{147} u^3 u_2^2,
\]
which is also only admitted by the UNSO-KdV equation (2.24).

The results obtained by using the Maple procedure CONSLAW are listed in the next table

<table>
<thead>
<tr>
<th>Rank (ρ)</th>
<th>Parameter constraints</th>
<th>N</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>{c = 2(b + d)}</td>
<td>1</td>
<td>0.360</td>
</tr>
<tr>
<td>4</td>
<td>{c = b + 3d, e = 5f - 10}</td>
<td>1</td>
<td>0.406</td>
</tr>
<tr>
<td>6</td>
<td>{d = \frac{45 + 14 + 3e - 45 + 10f^2 - 2ef}{2}, \ a = \frac{28f + 209 + 130f^2 - 72ef + 10f + 130f^2 - 10f^2}{130f^2 - 72ef - 54e - 180c - 81} }</td>
<td>2</td>
<td>0.469</td>
</tr>
<tr>
<td>8</td>
<td>{a = \frac{196 + 80f + 85f^2 - 90f - 10f^2}{2}, b = \frac{-9f + 2f^2 + 4f + 14f}{42}, d = \frac{2f - 9f - 2f^2}{42} }</td>
<td>3</td>
<td>0.783</td>
</tr>
<tr>
<td>10</td>
<td>{b = \frac{5}{14}, d = \frac{5}{14}, e = 5, f = 3}</td>
<td>7</td>
<td>3.5</td>
</tr>
<tr>
<td>12</td>
<td>{a = \frac{5}{14}, b = \frac{5}{14}, c = \frac{5}{14}, d = \frac{5}{14}, e = 5, f = 3}</td>
<td>10</td>
<td>8.389</td>
</tr>
<tr>
<td>14</td>
<td>Same as the condition of rank 12</td>
<td>14</td>
<td>37.502</td>
</tr>
<tr>
<td>16</td>
<td>Same as the condition of rank 10</td>
<td>18</td>
<td>752.986</td>
</tr>
<tr>
<td>20</td>
<td>Same as the condition of rank 12</td>
<td>20</td>
<td>1652.171</td>
</tr>
</tbody>
</table>

where the values in each line stand for the rank of the conserved densities ρ; the parameters constraints under which the conserved densities existing; N, the terms included in the conserved densities ρ and the CPU time of only computing the conserved densities for the fixed Rank (ρ)’s. Up to now, we have computed the conserved densities of ranks 2–20 due to the limitation of the physical amount of memory in the computer. The results show that: (i) The GSO-KdV equation possesses infinitely many CLaws provided the parameters satisfying various conditions. (ii) The seventh order Lax equation possesses infinitely many polynomial type CLaws in each level, whereas, the SK-Ito equation has infinitely many CLaws with gap. Recall that the completely integrable KdV equations possess infinite sequences of conserved densities, and this property is used as a criterion to characterize complete integrability. Hence, we conjure that the UNSO-KdV equation is integrable. Further more, the fact that the SK-Ito equation possesses solitary wave solution and soliton solution motivates us to do the same research on the UNSO-KdV type equation.

3. Solitary wave solution and soliton solution for the UNSO-KdV equation

As mentioned before, the UNSO-KdV type equation has not been studied with regards to its solitary wave solution and soliton solution. To construct the solitary wave solution, introducing a nonlinear transformation of dependent variable [9],

\[
u(x, t) = r + ρ[ln(f(x, t))]_{xx},
\]
(3.1)
where $p$ and $r$ are real parameters to be determined ($p \neq 0$), which allows us to transform Eq. (2.2) into a ninth degree homogeneous equation, denoted by $Q$, consisting of 141 monomials in $f(x,t)$ and its derivatives. Then, we seek a solution of type

$$f(x,t) = 1 + \exp(\xi), \quad \xi = kx + wt + \zeta_0,$$

(3.2)

where $k$ and $w$ are parameters to be determined ($k \neq 0$), $\zeta_0$ is an arbitrary constant. Substituting (3.2) into the homogeneous equation $Q$ and equating the coefficients of different powers of $\exp(\xi)$ to zero, one obtains a nonlinear system of eight algebraic equations to determine the possible relations between the parameters $r$, $p$, $k$ and $w$. Solving the nonlinear system of algebraic equations yields

$$\left\{ \begin{array}{l}
k = k, \quad p = 21, \quad r = -\frac{7}{4} k^2, \quad w = \frac{1}{48} k^3 \end{array} \right\}.$$

Then the solution is

$$u(x,t) = -\frac{7}{4} k^2 \left\{ 1 - 12 \frac{\exp(\xi)}{[1 + \exp(\xi)]^2} \right\},$$

which can also be written as

$$u(x,t) = -\frac{7}{4} k^2 \left[ 1 - 3 \text{sech}^2 \left( \frac{\xi}{2} \right) \right],$$

where $\xi = k(x + \frac{1}{48} k^6 t) + \zeta_0$. Obviously, this solitary wave solution can also be obtained using the methods presented in [10–15].

Alternatively, if assuming the solution is of the form

$$f(x,t) = s + \exp(\xi) + \exp(2\xi)$$

and carrying through the same procedure as before, we first get

$$\left\{ \begin{array}{l}s = s, \quad k = k, \quad p = 21, \quad r = -\frac{7}{4} k^2(6s - 1), \quad w = \frac{k^2(4096s^3 + 960s^2 + 264s - 1)}{48(4s - 1)^3} \end{array} \right\}.$$

Then the solution is

$$u(x,t) = r + \frac{21 k^2 \exp(\xi) [s + 4s \exp(\xi) + \exp(2\xi)]}{[s + \exp(\xi) + \exp(2\xi)]^2},$$

which can also be written as

$$u(x,t) = r + 21 k^2 \frac{1 + 2 \text{csch}(\xi)}{[2 + \text{csch}(\xi)]^2},$$

(3.4)

where

$$\tilde{\xi} = kx + \frac{k^2(4096s^3 + 960s^2 + 264s - 1)}{48(4s - 1)^3} t - \ln \sqrt{s} + \zeta_0.$$

The solution (3.4) is a new solitary wave solution for Eq. (2.24), for which, if setting $s = 1/16$, we get $r = 0$ and $w = -k^3$. Then Eq. (3.4) is reduced to

$$u(x,t) = 21 k^2 \frac{1 + 2 \text{csch}(\tilde{\xi})}{[2 + \text{csch}(\tilde{\xi})]^2},$$

(3.5)

where $\tilde{\xi} = k(x - k^6 t) + 2 \ln 2 + \zeta_0$. The solution (3.5) vanishing at $\xi = \pm \infty$ is a new soliton solution for the unnamed Eq. (2.24) and cannot be obtained by using the methods presented in [10–15].

4. Conclusion

In summary, rich higher order polynomial type CLaws for the Lax, SK-Ito, UNSO-KdV equations as well as for the GSO-KdV equation are obtained. By introducing useful nonlinear transformation, the soliton as well as the solitary
wave solution for the UNSO-KdV type equation are constructed, which, to our knowledge, are new and first reported here.

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References