Multi-component WKI equations and their conservation laws

Changzheng Qu\textsuperscript{a,b,*}, Ruoxia Yao\textsuperscript{c,d}, Ruochen Liu\textsuperscript{a}

\textsuperscript{a} Department of Mathematics, Northwest University, Xi'an 710069, PR China
\textsuperscript{b} Center for Nonlinear Studies, Northwest University, Xi'an 710069, PR China
\textsuperscript{c} Department of Computer Sciences, East China Normal University, Shanghai 200062, PR China
\textsuperscript{d} Department of Computer Sciences, Weinan Teacher's College, Weinan 715500 PR China

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Abstract

In this Letter, a two-component WKI equation is obtained by using the fact that when curvature and torsion of a space curve satisfy the vector modified KdV equation, a graph of the curve satisfies the two-component WKI equation, which is a natural generalization to the WKI equation. It is shown that the two-component WKI equation can be solved in terms of the extended WKI scheme, and it admits an infinite number of conservation laws. In the same vein, an \textit{n}-component generalization to the WKI equation is proposed.

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1. Introduction

In this Letter, we intend to derive a multi-component generalization of the WKI equation \cite{1} from the viewpoint of curve motions in Euclidean space. The two-component WKI equation is shown to be associated to the vector modified mKdV equation

\[ \vec{k}_t + \vec{k}_{xxx} + \frac{3}{2} \vec{k}^2 \vec{k}_x = 0, \]

(1)

where \( \vec{k} = (k_1, k_2) \). When \( k_2 = 0 \), it is reduced to the celebrated mKdV equation

\[ k_t + k_{xxx} + \frac{3}{2} k^2 k_x = 0. \]

(2)

* Corresponding author.
E-mail address: qu_changzheng@hotmail.com (C. Qu).
It has been shown that Eq. (2) arises from an inextensible plane curves in Euclidean space [3,4], the plane curve flow is governed by
\[ \gamma_t = -k_s n - \frac{1}{2} k_s t, \]  
(3)
where \( s \) is the arc-length of the curve. Suppose this flow can be expressed as the graph \((x, u(x, t))\) of some function \( u \) on the \( x \)-axis. Using the fact that the normal speed of the curve \( \gamma, u_t/\left(1 + u_x^2\right)^{1/2}, \) is given by \(-k_s\), one finds that \( u \) satisfies [4]
\[ u_t = \left[ \frac{u_{xx}}{(1 + u_x^2)^{1/2}} \right]_x. \]  
(4)
It is nothing but the WKI equation [1]. In fact, it turns out that the integrability of (4) is established by Wadati, Konno and Ichikawa [5] who showed that it is the compatibility condition of a certain WKI scheme of inverse scattering transformation. This WKI-scheme for \( u \) is connected to the AKNS-scheme for \( k \) by a gauge transformation explicitly displayed in [6].

It has been shown that the vector mKdV equation
\[ \vec{\gamma}_t = \vec{k}_{xx} + \frac{3}{2} \vec{k}^2 \vec{k}_s, \]  
(5)
is integrable in the sense that it has infinitely many higher order Lie Backlund symmetries [7], where \( \vec{k} = (k_1, k_2, \ldots, k_n) \) is a vector with \( n \) components. Indeed, it admits the following recursion operator which maps a symmetry to a new symmetry [8]
\[ R = D^2_2 + |\vec{k}|^2 + \vec{k}_s D^{-1}_2 (\vec{k}, \cdot) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (J_{ij} \vec{k}) D^{-1}_2 (J_{ij} \vec{k}_s, \cdot), \]
where \( J_{ij} \) are antisymmetric matrices with nonzero entry of \((i, j)\) being 1 if \( i < j \), that is \((J_{ij})_{kl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \). Two questions arise: first, whether the system (5) can be obtained naturally from space curve motions in the \((n + 1)\)-dimensional Euclidean space or not, this problem has been discussed by several authors [3,10,12]; second, what is the multi-component generalization to the WKI equation?

The purpose of this Letter is to answer the second question. The outline of this Letter is as follows: in Section 2, we obtain the two-component WKI equation by considering motion of a nonextensible space curve in Euclidean space. A form of \( n \)-component generalization of the WKI equation is also proposed. In Section 3, we study the integrability of the two-component WKI equation and show that the two-component WKI equation admits an infinite number of conservation laws. A concluding remarks to this work is given in Section 4.

2. The two-component WKI equation

It has been known for a long time that motion of curves are closely related to integrable equations [2–4, 9–23]. We now consider motion of a space curve \( \gamma \) in three-dimensional Euclidean space \( \mathbb{R}^3 \), this problem has been considered by several authors, see Refs. [3,10,12]. The curve motion is specified by
\[ \gamma_t = U \mathbf{n} + V \mathbf{b} + W \mathbf{t}, \]  
(6)
where \( \mathbf{t}, \mathbf{b} \) and \( \mathbf{n} \) are respectively the tangent, binormal and normal vectors, and \( U, V, \) and \( W \) are the corresponding normal, binormal and tangent velocities, and they depend on the curvature and torsion of the curve and their derivatives with respect to the arc-length \( s \). The Serret–Frenet formula in \( \mathbb{R}^3 \) reads
\[ \frac{\partial \mathbf{t}}{\partial s} = k \mathbf{n}, \quad \frac{\partial \mathbf{n}}{\partial s} = -k \mathbf{t} + \tau \mathbf{b}, \quad \frac{\partial \mathbf{b}}{\partial s} = -\tau \mathbf{n}. \]  
(7)
As the curve evolves according to (6), after using the Serret–Frenet formula we obtain the time evolution of some relevant geometric quantities

\[ \ddot{\mathbf{t}} = \left( \frac{\partial V}{\partial s} - \frac{\tau}{k} \mathbf{U} \right) \mathbf{n} + \left( \frac{\partial V}{\partial s} + \frac{\tau}{k} \right) \mathbf{b}, \]

\[ \ddot{\mathbf{n}} = - \left( \frac{\partial V}{\partial s} - \frac{\tau}{k} \mathbf{U} \right) \mathbf{t} + \left[ \frac{1}{k} \frac{\partial}{\partial s} \left( \frac{\partial V}{\partial s} + \frac{\tau}{k} \mathbf{U} \right) + \frac{\tau}{k} \left( \frac{\partial U}{\partial s} - \tau V + kW \right) \right] \mathbf{b}, \]

\[ \ddot{\mathbf{b}} = - \left( \frac{\partial V}{\partial s} + \tau U \right) \mathbf{t} - \left[ \frac{1}{k} \frac{\partial}{\partial s} \left( \frac{\partial V}{\partial s} + \tau U \right) + \frac{\tau}{k} \left( \frac{\partial U}{\partial s} - \tau V + kW \right) \right] \mathbf{n}, \]

\[ \ddot{g} = g \left( \frac{\partial W}{\partial s} - kU \right), \]  \hspace{1cm} (8)

where \( \ddot{\gamma} = \partial(\gamma)/\partial t \), \( g \) stands for the metric of the curve, defined by \( ds = g \partial p \). \( p \) is an arbitrary arc-length. The compatibility condition between (7) and (8) implies \( k \) and \( \tau \) satisfying the system

\[ \frac{\partial k}{\partial t} = \frac{\partial^2 U}{\partial s^2} + (k^2 - \tau^2)U + \frac{\partial k}{\partial s} \int kU ds' - 2\tau \frac{\partial V}{\partial s} - \frac{\partial \tau}{\partial s} V, \]

\[ \frac{\partial \tau}{\partial t} = \frac{\partial}{\partial s} \left[ \frac{1}{k} \frac{\partial}{\partial s} \left( \frac{\partial V}{\partial s} + \tau U \right) + \frac{\tau}{k} \left( \frac{\partial U}{\partial s} - \tau V + kW \right) \right] + k\tau U + k \frac{\partial V}{\partial s}. \]  \hspace{1cm} (9)

Now, we choose \( \hat{k} = (k_1(s, t), k_2(s, t)) \), \( k_1 = k \cos \alpha \), \( k_2 = k \sin \alpha \), \( \alpha = \int^s \tau(s', t) ds' \), and \( U = -k_1, \ V = -k\tau \), then \( k \) and \( \tau \) satisfy

\[ k_t = -k_{sss} - \frac{3}{2} k^2 k_s - \frac{3}{2} k^2 k_t \equiv -J_1 - \frac{3}{2} k^2 k_s, \]

\[ \alpha_t = -\frac{3}{k} \frac{\tau}{k_s} - \frac{3}{k} \frac{k_1 \tau}{k} - \frac{k_s \tau}{k} + \frac{\tau}{k} - \frac{3}{2} k^2 \tau \equiv -\frac{1}{k} J_2 - \frac{3}{2} k^2 \tau, \]  \hspace{1cm} (10)

where

\[ J_1 = k_s s^3 - k \tau s^2, \quad J_2 = 3k_s s^3 + k_s s^2 + k (s_s^3 - \tau^2). \]

A direct computation yields

\[ k_{1,s} = k_s \cos \alpha - k \tau \sin \alpha, \quad k_{2,s} = k_s \sin \alpha + k \tau \cos \alpha, \]

\[ k_{1,sss} = J_1 \cos \alpha - J_2 \sin \alpha, \quad k_{2,sss} = J_1 \sin \alpha + J_2 \cos \alpha. \]  \hspace{1cm} (11)

Using (11), we obtain

\[ k_{1,t} = k_s \cos \alpha - k \tau \sin \alpha = - \left( J_1 + \frac{3}{2} k^2 k_s \right) \cos \alpha - k \sin \alpha \left( \frac{1}{k} J_2 - \frac{3}{2} k^2 \tau \right) = -k_{1,sss} - \frac{3}{2} k^2 k_{1,s}, \]

\[ k_{2,t} = k_s \sin \alpha + k \tau \sin \alpha = - \left( J_1 + \frac{3}{2} k^2 k_s \right) \sin \alpha + k \cos \alpha \left( \frac{1}{k} J_2 - \frac{3}{2} k^2 \tau \right) = -k_{2,sss} - \frac{3}{2} k^2 k_{2,s}, \]

which shows that \( \hat{k} \) satisfies the vector mKdV equation (1).

The corresponding curve motion flow is

\[ \gamma_t = -k_s \mathbf{n} - k \mathbf{b} - \frac{1}{2} k^2 \mathbf{t}. \]  \hspace{1cm} (12)

To derive the two-component WKI equation, we denote the curve in terms of its graph

\[ \gamma = (x, u(x, t), v(x, t)). \]  \hspace{1cm} (13)
Using the graph, its relevant geometric quantities are computed as following:

$$ds = g \, dx, \quad t = (1, u_x, v_x)/g,$$

$$n = \left( -\frac{1}{2} \frac{g^2}{\tilde{g}} (1 + v^2) u_x + u_x v_x v_{xx}, (1 + u^2) v_{xx} - u_x v_x u_{xx} \right)/\left( (\tilde{g} g) \right),$$

$$b = (u_x v_{xx} - v_x u_{xx}, -v_x, u_{xx})/\tilde{g}, \quad k = g^{3/2}, \quad \tau = \frac{u_{xx} v_{xxx} - v_{xx} u_{xxx}}{g^2}, \quad (14)$$

where \( g = \sqrt{1 + u^2 + v^2}, \ \tilde{g} = \sqrt{u^2 + v^2 + (u_x v_x - v_x u_x)^2} \). Multiplying (12) respectively by \( n \) and \( b \) and using the expressions (14), we obtain

$$v_x u_t - u_x v_t = \frac{u_{xx} v_{xxx} - v_{xx} u_{xxx}}{g^3},$$

$$\left[ (1 + v^2) u_{xx} - u_x v_x v_{xx} \right] u_t + \left[ (1 + u^2) v_{xx} - u_x v_x u_{xx} \right] v_t = -\tilde{g} k_x. \quad (15)$$

Solving system (15) for \( u_t \) and \( v_t \), we obtain the two-component WKI equation given by

$$u_t + \left[ \frac{u_{xx}}{(1 + u_x^2 + v_x^2)^{3/2}} \right] = 0, \quad v_t + \left[ \frac{v_{xx}}{(1 + u_x^2 + v_x^2)^{3/2}} \right] = 0. \quad (16)$$

It is interesting to note that the system (16) is a natural generalization of the WKI equation (4) and is reduced to (4) when \( v = 0 \). In the next section, we will show that the 2-component WKI equations (16) is integrable and can be obtained from the WKI scheme [5]. In the same vein, the \( n \)-component WKI equation

$$u_{1t} + \left[ \frac{u_{1xx}}{(1 + \sum_{i=1}^{n} u_{ix}^2)^{3/2}} \right] = 0, \quad u_{2t} + \left[ \frac{u_{2xx}}{(1 + \sum_{i=1}^{n} u_{ix}^2)^{3/2}} \right] = 0, \quad \ldots$$

$$u_{nt} + \left[ \frac{u_{ntxx}}{(1 + \sum_{i=1}^{n} u_{ix}^2)^{3/2}} \right] = 0, \quad (17)$$

can be obtained from the motion of a nonextensible curve in the \((n + 1)\)-dimensional Euclidean space.

3. Conservation laws

In this section, we study the integrability and the existence of an infinite number of conservation laws of (16). We consider the following eigenvalue problem

$$\psi_{1x} + i \lambda \psi_1 = -\lambda (u_x + iv_x) \psi_2, \quad \psi_{2x} - i \lambda \psi_2 = \lambda (u_x - iv_x) \psi_1. \quad (18)$$

The time dependence of the eigenfunctions is chosen to be

$$\psi_{1t} = A \psi_1 + B \psi_2, \quad \psi_{2t} = C \psi_1 - A \psi_2, \quad (19)$$

where

$$A = -\frac{4i}{g} \lambda^3 + \frac{2i(u_x v_{xx} - v_x u_{xx})}{g^3} \lambda^2, \quad B = \frac{4(u_x + i v_x)}{g} \lambda^3 - \frac{2i(u_x + i v_x^2)}{g^3} \lambda^2 + \left( \frac{u_{xx} + i v_{xx}}{g^3} \right) \lambda,$$

$$C = -\frac{4(u_x - i v_x)}{g} \lambda^3 - \frac{2i u_{xx} - i v_{xx}}{g^3} \lambda^2 + \left( \frac{u_{xx} - i v_{xx}}{g^3} \right) \lambda.$$

The compatibility condition between (18) and (19) gives the two-component WKI equation.
Introducing the variable
\[ H = \psi_2/\psi_1, \]
then the system (18) is written as
\[
\frac{\partial}{\partial t}[(u_x + iv_x)H] = \frac{1}{\lambda} \frac{\partial}{\partial x}(A + BH),
\]
\[
2i\lambda[(u_x + iv_x)H] = \lambda(u_x^2 + v_x^2) + \lambda((u_x + iv_x)H)^2 + (u_x + iv_x)\frac{\partial}{\partial x}H,
\]
(20) after using (19). Expanding
\[
H = \sum_{n=0}^{+\infty} h_n\lambda^{-n},
\]
we obtain an infinite number of conservation laws, and a recursion formula for the conserved densities \( h_n \):
\[
2i h_n = (u_x^2 + v_x^2)\delta_{n,0} + \sum_{l=0}^{n} h_l h_{n-l} + (u_x + iv_x)\frac{\partial}{\partial x}[h_{n-1}/(u_x + iv_x)],
\]
where \( \delta_{n,0} = 1 \), if \( n = 1 \) and \( \delta_{n,0} = 0 \), if \( n \neq 0 \). The first seven conserved densities are given by
\[
p_1 = u,
p_2 = v,
p_3 = \sqrt{1 + u_x^2 + v_x^2},
p_4 = (g^{-1} - 1)\ln[(u_x^2 + v_x^2)]_x + 2\ln|g|_x,
p_5 = (g^{-1} - 1)u_xv_{xx} - u_xv_x,
p_6 = \frac{1}{8}g^{-3}(u_x^2 + v_x^2) - \frac{1}{8}g^{-5}(u_x u_{xx} + v_x v_{xx}) - \frac{1}{8}g^{-3}(u_x u_{xx} + v_x v_{xx})g(u_x^2 - v_x^2)]_x,
p_7 = [g^{-2}(1 - g)(u_x u_{xx} - u_x x v_{xx})]_x.
\]
One can show that the two-component mKdV equation (2) also admits an infinite number of conservation laws, and the first several conserved densities are
\[
q_1 = k_1^2 + k_2^2,
q_2 = k_1 k_{2x},
q_3 = (k_1^2 + k_2^2)^2 - 4(k_1^2 k_2^2) + k_1^2 k_{2x}^2 - 3k_2^2 k_{1xx},
q_4 = 3k_1^2 k_2^2 k_{2x} + k_1^3 k_{2x} - 3k_2^2 k_{1xx},
q_5 = (k_1^2 + k_2^2)^3 - 12(k_2^2 k_{1x}^2 + k_1^2 k_{2x}^2) - 20(k_2^2 k_{1x}^2 + k_1^2 k_{2x}^2) + 8(k_1^2 k_{1xx} + k_2^2 k_{2xx}) - 16k_1 k_2 k_{1x} k_{2x},
q_6 = 5(k_1 k_{1x} k_{2x} + k_1 k_{1x} k_{2x} - k_1 k_{2x} k_{2x}) - 5k_1^2 k_2^2 k_{2x} + k_{1xx} k_{2xx} + \frac{15}{8}(k_1^4 k_{2} k_{1x} - k_1 k_2^4 k_{2x}),
q_7 = 36(k_1^2 k_{2x}^2 + k_1^2 k_{1xx}) + (k_1^2 + k_2^2)^4 + 84(k_1^2 k_{1x}^2 + k_1 k_2^2 k_{2x}) + \frac{288}{5}k_1 k_{1x} k_{2x} k_{2x} + 96k_1^3 k_{1x} k_{2x} + 120k_1^2 k_{2x}^2 - \frac{252}{5}k_1^2 k_{1x}^2 + \frac{504}{5}(k_1^2 k_{2x}^2 + k_1^2 k_{1xx}) + 72(k_1^2 k_{2x}^2 + k_1^2 k_{1xx}) + \frac{216}{5}(k_1^2 k_{1xx} + k_1^2 k_{2xx}) + 864(k_2 k_{1x} k_{2x} + k_1 k_{2x} k_{1xx}) - 48k_1^2 k_{2x}^2 - 24k_1^2 k_{2xx} + \frac{72}{5}k_1^2 k_{2x}^2.
4. Concluding remarks

In this Letter we have obtained the multi-component WK I equation from the motion of space curves in Euclidean space. It is interesting to compare it with that the WK I equation can be obtained from the motion of plane curves in Euclidean space. We have shown that the two-component WK I equation can be solved in terms of the WKI scheme and it admits an infinite number of conservation laws. In [23,24], it has been shown that the \( K(m, m + 2) \) model

\[
k_t = (k^m)_{xxx} + \frac{m}{m + 1} (k^{m+2})_x,
\]

arises from the plane curve motion in certain Klein geometries, and the \( K(m, m + 2) \) model is geometrically equivalent to the generalized WKI equation [25]

\[
u_t = \left[ \left( \frac{u_{xx}}{1 + u_x^2} \right)^m \right]_x.
\]

Similarly, we can obtain the 2-component \( K(m, m + 2) \) models

\[
u_t = \left[ \left( u^m \right)_{xxx} + m \left[ u^{m-1} (u^2 + v^2) + \frac{1}{m + 1} (u^{m+1} + v^{m+1}) \right] u_x,\right.
\]

\[
v_t = \left[ \left( v^m \right)_{xxx} + m \left[ v^{m-1} (u^2 + v^2) + \frac{1}{m + 1} (u^{m+1} + v^{m+1}) \right] v_x,\right.
\]

and \((n - 1)\)-component \( K(m, m + 2) \) models respectively from the motion of curves in the 3-dimensional and \( n \)-dimensional Euclidean space.

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